SOME FURTHER IDENTITIES FOR THE GENERALIZED FIBONACCI SEQUENCE $\{H_n\}$

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1. INTRODUCTION

In this paper we are concerned with developing and establishing further identities for the generalized Fibonacci sequence $\{H_n\}$, with particular emphasis on summation properties. First we obtain a number of power identities by substitution into some known identities and then we establish a number of summation identities. Next we proceed to derive some further summation identities involving the fourth power of generalized Fibonacci numbers $\{H_n\}$ from a consideration of the ordinary Pascal triangle. Finally, we arrive at some additional summation identities by applying standard difference equation theory to the sequence $\{H_n\}$. Notation and definitions of Walton and Horadam [9] are assumed.

2. POWER IDENTITIES FOR THE SEQUENCE $\{H_n\}$

In this section a number of new power identities for the generalized Fibonacci numbers $\{H_n\}$ have been obtained by following the reasoning of Zeitlin [10], for similar identities relating to the ordinary Fibonacci sequence $\{F_n\}$.

Use will be made of identities (11) and (12) of Horadam [6], viz.,

$$H_nH_{n+2} - H_{n+1}^2 = (-1)^{n+1}d \tag{2.1}$$

$$H_{m+h}H_{m+k} - H_mH_{m+h+k} = (-1)^{m+2h}dF_hF_k \tag{2.2}$$

(where we have substituted $n = m+h$, $h = s$ and $k = r+s+1$), and the identity

$$H_{k+1}H_{m-k} + H_kH_{m-k-1} = (2p-q)H_m - dF_m , \tag{2.3}$$

where the right-hand side of (2.3) is derived from (9) of Horadam [6].

Re-writing (2.1) in the form

$$H_n^2 - H_{n+1}^2 = (-1)^{n+1}d - H_nH_{n+1} \tag{2.4}$$

yields

$$H_{n+1}^4 + H_n^4 = (H_n^2 - H_{n+1}^2)^2 + 2H_n^2H_{n+1}^2 = d^2 + 2(-1)^ndH_nH_{n+1} + 3H_n^2H_{n+1}^2 \tag{2.5}$$

$$\begin{aligned} -2H_{n+1}^3H_n - H_{n+1}^2H_n^2 + 2H_{n+1}H_n^3 &= 2H_nH_{n+1}[(-1)^{n+1}d - H_nH_{n+1}] - H_n^2H_{n+1}^2 \\ &= -2(-1)^ndH_nH_{n+1} - 3H_n^2H_{n+1}^2 . \end{aligned} \tag{2.6}$$

Adding (2.5) and (2.6) gives

$$H_{n+1}^4 - 2H_{n+1}^3H_n - H_{n+1}^2H_n^2 + 2H_{n+1}H_n^3 + H_n^4 = d^2 . \tag{2.7}$$

If we now substitute the identities

*Part of the substance of an M.Sc. thesis presented to the University of New England in 1968.

$$\text{(2.8)} \qquad \begin{cases} H_{n+4} = 3H_{n+1} + 2H_n \\ H_{n+3} = 2H_{n+1} + H_n \\ H_{n+2} = H_{n+1} + H_n \end{cases}$$

into the expression

$$H_{n+4}^4 - 4H_{n+3}^4 - 19H_{n+2}^4 - 4H_{n+1}^4 + H_n^4$$

we have -6 times the left-hand side of (2.7), *i.e.,*

$$\text{(2.9)} \qquad H_{n+4}^4 - 4H_{n+3}^4 - 19H_{n+2}^4 - 4H_{n+1}^4 + H_n^4 = -6d^2 .$$

Re-arranging (2.9) and substituting $n = n + 1$ yields

$$\text{(2.10)} \qquad H_{n+5}^4 = 4H_{n+4}^4 + 19H_{n+3}^4 + 4H_{n+2}^4 - H_{n+1}^4 - 6d^2$$

so that substitution for $-6d^2$ from (2.9) gives

$$\text{(2.11)} \qquad H_{n+5}^4 = 5H_{n+4}^4 + 15H_{n+3}^4 - 15H_{n+2}^4 - 5H_{n+1}^4 + H_n^4 .$$

We note here that (2.9) is a verification of (4.6) of Zeitlin [11].

If we now let $V_n = H_{n+1}^4 - H_n^4$, we may re-write (2.9) in the form

$$\text{(2.12)} \qquad V_{k+3} - 3V_{k+2} - 22V_{k+1} - 26V_k - 25H_k^4 = -6d^2 ,$$

where

$$\sum_{k=0}^{n} V_{k+j} = H_{n+j+1}^4 - H_j^4 .$$

Summing both sides of (2.12) over k, where $k = 0, 1, \cdots, n$, gives

$$\text{(2.13)} \qquad 25 \sum_{k=0}^{n} H_k^4 = H_{n+4}^4 - 3H_{n+3}^4 - 22H_{n+2}^4 - 26H_{n+1}^4 + 6(n+1)d^2 + \delta ,$$

where

$$\delta = 9p^4 - 20p^3q - 6p^2q^2 + 4pq^3 + 28q^4 .$$

($\delta = 9$ for the Fibonacci numbers $\{F_n\}$.)

Substituting for H_{n+4}^4 in (2.13) by using (2.9) gives

$$\text{(2.14)} \qquad 25 \sum_{k=0}^{n} H_k^4 = H_{n+3}^4 - 3H_{n+2}^4 - 22H_{n+1}^4 - H_n^4 + 6nd^2 + \delta$$

which yields the obvious result

$$\text{(2.15)} \qquad H_{n+3}^4 - 3H_{n+2}^4 - 22H_{n+1}^4 - H_n^4 + 6nd^2 + \delta' \equiv 0 \mod 25 ,$$

where

$$\delta' = 9p^4 - 20p^3q - 6p^2q^2 + 4pq^3 + 3q^4 .$$

($\delta' = 9$ for the Fibonacci numbers $\{F_n\}$.)

Multiplying (2.11) by $(-1)^{n+5}$ and replacing n by k gives

$$\text{(2.16)} \qquad W_{k+4} + 6W_{k+3} - 9W_{k+2} - 24W_{k+1} - 19W_k = 18(-1)^k H_k^4 ,$$

where

$$\text{(2.17)} \qquad W_n = (-1)^{n+1}H_{n+1}^4 - (-1)^n H_n^4 .$$

Summing over both sides of (2.16) for $k = 0, 1, \cdots, n$, and using

$$\text{(2.18)} \qquad \sum_{k=0}^{n} W_{k+j} = (-1)^{n+j+1}H_{n+j+1}^4 - (-1)^j H_j^4$$

gives

(2.19) $$18 \sum_{k=0}^{n} (-1)^k H_k^4 = (-1)^n [-H_{n+5}^4 + 6H_{n+4}^4 + 9H_{n+3}^4 - 24H_{n+2}^4 + 19H_{n+1}^4] + 6\epsilon$$

$$= (-1)^n [H_{n+4}^4 - 6H_{n+3}^4 - 9H_{n+2}^4 + 24H_{n+1}^4 - H_n^4] + 6\epsilon \quad \text{by (2.11)}$$

$$= (-1)^n [-2H_{n+3}^4 + 10H_{n+2}^4 + 28H_{n+1}^4 - 2H_n^4 - 6d^2] + 6\epsilon \quad \text{by (2.9)},$$

where

$$\epsilon = 2p^3q - 3p^2q^2 - 2pq^3 + 3q^4 \left(= q(2p^3 - 3p^2q - 2pq^2 + 3q^3) \right).$$

($\epsilon = 0$ for the Fibonacci numbers $\{F_n\}$.)

Therefore, on using (2.11), we have

(2.20) $$18 \sum_{k=0}^{n} (-1)^k H_k^4 = (-1)^n [H_{n+4}^4 - 6H_{n+3}^4 - 9H_{n+2}^4 + 24H_{n+1}^4 - H_n^4] + 6\epsilon$$

$$= 2\left\{(-1)^n [-H_{n+3}^4 + 5H_{n+2}^4 + 14H_{n+1}^4 - H_n^4 - 3d^2] + 3\epsilon\right\}$$

on using (2.9). Now (2.20) implies that

(2.21) $$H_{n+4}^4 - 6H_{n+3}^4 - 9H_{n+2}^4 + 24H_{n+1}^4 - H_n^4 \equiv 0 \mod 6$$

from which we conclude that

(2.22) $$H_{n+4}^4 - 9H_{n+2}^4 - H_n^4 \equiv 0 \mod 6$$

so that

(2.23) $$H_{n+4}^4 - H_n^4 \equiv 0 \mod 3 \quad .$$

We will now use the identity

(2.24) $$H_{k+1}H_{k+2}H_{k+4}H_{k+5} = H_{k+3}^4 - d^2$$

(which is a generalization of an identity for the sequence $\{F_n\}$ stated by Gelin and proved by Cesàro — see Dickson [2]) to establish the two results

(2.25) $$25 \sum_{k=0}^{n} H_{k+1}H_{k+2}H_{k+4}H_{k+5} = 26H_{n+3}^4 + 22H_{n+2}^4 + 3H_{n+1}^4 - H_n^4 - 19nd^2 - 25d^2 + \delta - 50t^2$$

(2.26) $$9 \sum_{k=0}^{m} (-1)^k H_{k+1}H_{k+2}H_{k+4}H_{k+5} = (-1)^m [-H_{m+6}^4 + 5H_{m+5}^4 + 14H_{m+4}^4 - H_{m+3}^4 - 3d^2] - 3\epsilon - 9d^2 g(m) + 18\gamma \;,$$

where

$$g(m) = \begin{cases} 0 & \text{if } m = 2n-1, \quad n = 1, 2, \cdots \\ 1 & \text{if } m = 2n, \quad n = 0, 1, \cdots \end{cases}$$

and

$$\begin{cases} \gamma = q^4 + 2q^3p + 3q^2p^2 + 2qp^3 \left(= q(q^3 + 2q^2p + 3qp^2 + 2q^3) \right) \\ t = p^2 + pq + q^2 \quad . \end{cases}$$

for the Fibonacci numbers $\{F_n\}$, $\gamma = 0$, $t = 1$.

Proof: Sun both sides of (2.24) with respect to k. Then

(2.27) $$25 \sum_{k=0}^{n} H_{k+1}H_{k+2}H_{k+4}H_{k+5} = 25 \sum_{k=0}^{n} H_{k+3}^4 - 25(n+1)d^2$$

(2.28) $$9 \sum_{k=0}^{m} (-1)^k H_{k+1}H_{k+2}H_{k+4}H_{k+5} = 9 \sum_{k=0}^{m} (-1)^k H_{k+3}^4 - 9d^2 g(m) \;,$$

where

$$g(m) = \sum_{k=0}^{m} (-1)^k \;.$$

Now,

$$\sum_{k=0}^{n} H_{k+3}^4 = \sum_{j=0}^{n+3} H_j^4 - 2t^2 ,$$

where

$$t = p^2 + pq + q^2 ,$$

so that on using (2.14), with n replaced by $n+3$, the right-hand side of (2.27) reduces to

$$H_{n+6}^4 - 3H_{n+5}^4 - 22H_{n+4}^4 - H_{n+3}^4 - 19nd^2 - 7d^2 + \delta - 50t^2$$

Eliminating H_{n+6}^4, H_{n+5}^4 and H_{n+4}^4 by using (2.9) gives (2.25). Since

$$\sum_{k=0}^{m} (-1)^k H_{k+3}^4 = -\sum_{j=0}^{m+3} (-1)^j H_j^4 + 2\gamma ,$$

where

$$\gamma = q^4 + 2q^3p + 3q^2p^2 + 2pq^3 ,$$

use of (2.20), where $m+3$ replaces n, and of (2.28) yields (2.26).

From (2.2) with $m = n-j$, $h = j$ and $k = 1$, we obtain

$$H_nH_{n-j+1} - H_{n-j}H_{n+1} = (-1)^{n+j} dF_jF_1 = (-1)^{n+j} dF_j . \tag{2.29}$$

Now

$$H_n = H_{n+2} - H_{n+1} ,$$

so that (2.29) simplifies to

$$H_{n+2}H_{n+1-j} - H_{n+1}H_{n+2-j} = (-1)^{n+j} dF_j . \tag{2.30}$$

From (2.3), with $m = 2n+4-j$ and $k = n+2$, we obtain

$$(2p-q)H_{2n+4-j} - dF_{2n+4-j} = H_{n+3}H_{n+2-j} + H_{n+2}H_{n+1-j} . \tag{2.31}$$

Substituting for $H_{n+2}H_{n+1-j}$ in (2.30) by means of (2.31) gives

$$\begin{aligned}(2p-q)H_{2n+4-j} - dF_{2n+4-j} &= H_{n+3}H_{n+2-j} + H_{n+1}H_{n+2-j} + (-1)^{n+j} dF_j \\ &= (pL_{n+3} + qL_{n+2})H_{n+2-j} + (-1)^{n+j} dF_j\end{aligned} \tag{2.32}$$

which may be written as

$$\begin{aligned}&(-1)^{j+1} H_{j+1} \left\{ (2p-q)H_{2n+4-j} - dF_{2n+4-j} \right\} \\ &\quad = (-1)^{j+1} (pL_{n+3} + qL_{n+2})H_{n+2-j}H_{j+1} + (-1)^{n+1} dH_{j+1}F_j .\end{aligned} \tag{2.33}$$

From (2.2) with $m = j+1$, $h = n+1-j$ and $k = n+2-j$, we obtain

$$H_{n+2}H_{n+3} - H_{j+1}H_{2n+4-j} = (-1)^{j+1} dF_{n+1-j}F_{n+2-j} \tag{2.34}$$

so that

$$(-1)^{j+1} H_{j+1}(2p-q)H_{2n+4-j} = (-1)^{j+1}(2p-q)H_{n+2}H_{n+3} - d(2p-q)F_{n+1-j}F_{n+2-j} . \tag{2.35}$$

Substituting (2.35) into (2.33) gives

$$\begin{aligned}&(2p-q)dF_{n+1-j}F_{n+2-j} + (-1)^{j+1}(pL_{n+3} + qL_{n+2}) \cdot H_{n+2-j}H_{j+1} + (-1)^{j+1} dH_{j+1}F_{2n+4-j} \\ &\quad + (-1)^{n+1} H_{j+1}F_j = (-1)^{j+1}(2p-q)H_{n+2}H_{n+3} .\end{aligned} \tag{2.36}$$

The following identities may be proved by induction:

(2.37) $$2\sum_{k=0}^{n}(-1)^k H_{m+3k} = (-1)^n H_{m+3n+1} + H_{m-2} \qquad (m = 2, 3, \cdots)$$

(2.38) $$3\sum_{k=0}^{n}(-1)^k H_{m+4k} = (-1)^n H_{m+4n+2} + H_{m-2} \qquad (m = 2, 3, \cdots)$$

(2.39) $$11\sum_{k=0}^{n}(-1)^k H_{m+5k} = (-1)^n [5H_{m+5n+1} + 2H_{m+5n}] + 4H_m - 5H_{m-1} \qquad (m = 1, 2, \cdots)$$

(2.40) $$4\sum_{k=0}^{n} H_k H_{2k+1} = H_{2n+3}H_n + H_{2n}H_{2n+3} - 2q^2$$

(2.41) $$3\sum_{k=0}^{n}(-1)^k H^2_{m+2k} = (-1)^n H_{m+2n}H_{m+2n+2} + H_m H_{m-2} \qquad (m = 2, 3, \cdots)$$

(2.42) $$7\sum_{k=0}^{n}(-1)^k H^2_{m+4k} = (-1)^n H_{m+4n}H_{m+4n+4} + H_m H_{m-4} \qquad (m = 4, 5, \cdots)$$

(2.43) $$2\sum_{k=0}^{n} H_{k+2}H^2_{k+1} = H_{n+3}H_{n+2}H_{n+1} - pq(p+q)$$

(2.44) $$2\sum_{k=0}^{n}(-1)^k H_k H^2_{k+1} = (-1)^n H_{n+2}H_{n+1}H_n + pq(p-q)\ .$$

Zeitlin [11] has also examined numerous power identities for the sequence $\{H_n\}$ as special cases of even power identities found for the generalized sequence $\{\omega_n\}$ used in Horadam [7], and earlier by Tagiuri (Dickson [2]).

As seen in Horadam [7], the generalized Fibonacci sequence $\{H_n\}$ is a particular case of generalized sequence $\{\omega_n\}$ for $a = q$, $b = p$, $r = 1$ and $s = -1$. Hence applying these results to (3.1), Theorem I, of Zeitlin [11] yields, for $n = 0, 1, \cdots$ (see (2.47) below):

(2.45) $$(-1)^{mrn}\sum_{k=0}^{2t}(-1)^{mrt}b_k^{(2t)}\left(-\frac{i}{2}\right)H^{2r}_{m(n+2t-k)+n_0} \qquad (i = \sqrt{-1}\ )$$

$$= (-1)^{rn_0+mt(4r-t-1)/2}\binom{2r}{r}(-5)^{t-r}d^r\prod_{k=1}^{t}F^2_{mk}\ .$$

However,

$$\begin{aligned}(-1)^{mt(4r-t-1)/2} &= (-1)^{2mtr-mt(t+1)/2}\\ &= (-1)^{2mtr-mt(t+1)+mt(t+1)/2}\\ &= (-1)^{mt(t+1)/2}\end{aligned}$$

since $2mtr$ and $mt(t+1)$* are always even. Hence, we may rewrite (2.45) as

*This result for $mt(t+1)$ may be easily verified by considering the table

m	t	$t+1$	$mt(t+1)$
odd	odd	even	even
even	even	odd	even

$$(2.46)\qquad (-1)^{mrn}\sum_{k=0}^{2t}(-1)^{mrt}b_k^{(2t)}\left(-\frac{i}{2}\right)H^{2r}_{m(n+2t-k)+n_0}$$

$$= (-1)^{rn_0+mt(t+1)/2}\binom{2r}{r}(-5)^{t-r}d^r\prod_{k=1}^{t}F^2_{mk}\ ,$$

where $n_0 = 0, 1, \cdots; m, t = 1, 2, \cdots, r = 0, 1, \cdots, t,$ and where the

$$b_k^{(2t)}\left(-\frac{i}{2}\right),\qquad k = 0, 1, \cdots, 2t,$$

are defined (as a special case of (2.9) of Zeitlin [11]) by

$$(2.47)\qquad \sum_{k=0}^{2t} b_k^{(2t)}\left(-\frac{i}{2}\right)y^{2t-k} = \prod_{k=1}^{t}(y^2-(-1)^{mk}L_{2mk}y+1)\ .$$

If we now consider $r = t = 1$ in (2.46) and then (2.47), then (2.46) reduces to

$$(2.48)\qquad (-1)^{mn}[H^2_{m(n+2)+n_0} - L_{2m}H^2_{m(n+1)+n_0} + H^2_{mn+n_0}] = 2(-1)^{m+n_0}dF^2_n\ .$$

on calculation. This corresponds to (4.5) of Zeitlin [11].

Similarly, we can obtain (4.6) to (4.16) of Zeitlin [11] by the correct substitutions into (2.46) and (2.47), where as already mentioned, (4.6) is our previous identity, (2.9). Identities (4.7) to (4.16) of Zeitlin should be noted for reference and comparison.

3. FOURTH POWER GENERALIZED FIBONACCI IDENTITIES

Hoggatt and Bicknell [5] have derived numerous identities involving the fourth power of Fibonacci numbers $\{F_n\}$ from Pascal's triangle.

By considering the same matrices S and U where $u_1 = H_0 = q$ and $u_2 = H_1 = p$, *i.e.*,

$$(3.1)\qquad S = \begin{bmatrix} 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 1 & 4\\ 0 & 0 & 1 & 3 & 6\\ 0 & 1 & 2 & 3 & 4\\ 1 & 1 & 1 & 1 & 1\end{bmatrix}$$

and $U = (a_{ij})$ is the column matrix defined by

$$(3.2)\qquad a_{i1} = \binom{4}{i-1}H_0^{5-i}H_1^{i-1}\ ,\qquad i = 1, 2, \cdots, 5,$$

the following identities for the fourth power of generalized Fibonacci numbers may easily be verified by proceeding as in Hoggatt and Bicknell [5]:

$$(3.3)\qquad \sum_{i=0}^{4n+1}(-1)^i\binom{4n+1}{i}H^4_{i+j} = 25^n(H^4_{2n+j} - H^4_{2n+j+1}) = A_j\quad \text{(say)}$$

$$(3.4)\qquad \sum_{i=0}^{4n+2}(-1)^i\binom{4n+2}{i}H^4_{i+j} = 25^n(H^4_{2n+j} - 2H^4_{2n+j+1} + H^4_{2n+j+2}) = A_j - A_{j+1}$$

$$(3.5)\qquad \sum_{i=0}^{4n+3}(-1)^i\binom{4n+3}{i}H^4_{i+j} = 25^n(H^4_{2n+j} - 3H^4_{2n+j+1} + 3H^4_{2n+j+2} - H^4_{2n+j+3}) = A_j - 2A_{j+1} + A_{j+2}$$

$$(3.6)\qquad \sum_{i=0}^{4n+4}(-1)^i\binom{4n+4}{i}H^4_{i+j} = 25^n(H^4_{2n+j} - 4H^4_{2n+j+1} + 6H^4_{2n+j+2} - 4H^2_{2n+j+3} + H^4_{2n+j+4})$$
$$= A_j - 3A_{j+1} + 3A_{j+2} - A_{j+3}\ .$$

Noting that the coefficients of the terms involving the A's on the right-hand side of the above equations are the first four rows of Pascal's triangle, we deduce the general identity

$$(3.7) \qquad \sum_{i=0}^{4n+k} (-1)^i \binom{4n+k}{i} H_{i+j}^4 = 25^n (H_{2n+j}^4 - (k-1)H_{2n+j+1}^4 + \cdots + (-1)^{k-1} H_{2n+j+k}^4)$$
$$= A_j - (k-1)A_{j+1} + \cdots + (-1)^{k-1} A_{j+k} .$$

Similarly, we have

$$(3.8) \qquad \sum_{i=0}^{4n+5} (-1)^i \binom{4n+5}{i} H_{i+j}^4 = 25^{n+1}(H_{2n+j+2}^4 - H_{2n+j+3}^4) = 25A_{j+2} ,$$

which results in the recurrence relation

$$(3.9) \qquad A_j - 4A_{j+1} + 6A_{j+2} - 4A_{j+3} + A_{j+4} = 25A_{j+2}$$

i.e.,

$$(3.10) \qquad A_j - 4A_{j+1} - 19A_{j+2} - 4A_{j+3} + A_{j+4} = 0$$

on equating (3.8) and (3.7) with $k = 5$. Defining

$$(3.11) \qquad G(j) = H_{n+j}^4 - 4H_{n+j+1}^4 - 19H_{n+j+2}^4 - 4H_{n+j+3}^4 + H_{n+j+4}^4$$

yields

$$(3.12) \qquad 25^n \{G(j) - G(j+1)\} = A_j - 4A_{j+1} - 19A_{j+2} - 4A_{j+3} + A_{j+4}$$
$$= 0 \quad \text{on using (3.10)} .$$

Hence, $G(j)$ is a constant.

When $n = j = 0$, (3.11) reduces to

$$(3.13) \qquad G(0) = -6d^2 ,$$

which leads to identity (2.9) which is in turn a generalization of a result due to Zeitlin [10] while also being a verification of a result due to Hoggatt and Bicknell [5] and also Zeitlin [11].

4. FURTHER GENERALIZED FIBONACCI IDENTITIES

In addition to the numerous identities of, say, Carlitz and Ferns [1], Iyer [4], Zietlin [10], [11], Subba Rao [8] and Hoggatt and Bicknell [5], Harris [3] has also listed many identities for the Fibonacci sequence $\{F_n\}$ which may be generalized to yield new identities for the generalized Fibonacci sequence $\{H_n\}$.

$$(4.1) \qquad \sum_{k=0}^{n} kH_k = nH_{n+2} - H_{n+3} + H_3$$

Proof: If

$$u_k \Delta v_k = \Delta(u_k v_k) - v_{k+1} \Delta u_k$$

(Δ is the difference operator) then

$$\sum_{k=0}^{n} u_k \Delta v_k = [u_k v_k]_0^{n+1} - \sum_{k=0}^{n} v_{k+1} \Delta u_k \quad .$$

Let $u_k = k$ and $\Delta v_k = H_k$. Then

$$\Delta u_k = 1 \quad \text{and} \quad v_k = \sum_{i=0}^{k-1} H_i = H_{k+1} - p .$$

Omitting the constant $-p$ from v_k , we find

$$\sum_{k=0}^{n} kH_k = [kH_{k+1}]_0^{n+1} - \sum_{k=0}^{n} 1\cdot H_{k+2} = (n+1)H_{n+2} - H_{n+4} - p - H_1 - H_0 = nH_{n+2} - H_{n+3} + (2p+q).$$

Using this technique, we also have the following identities:

$$\sum_{k=0}^{n} (-1)^k kH_k = (-1)^n (n+1)H_{n-1} + (-1)^{n-1} H_{n-2} - H_{-3} \tag{4.2}$$

$$\sum_{k=0}^{n} kH_{2k} = (n+1)H_{2n+1} - H_{2n+2} + H_0 \tag{4.3}$$

$$\sum_{k=0}^{n} kH_{2k+1} = (n+1)H_{2n+2} - H_{2n+3} + H_1 \tag{4.4}$$

$$\sum_{k=0}^{n} k^2 H_{2k} = (n^2+2)H_{2n+1} - (2n+1)H_{2n} - (2o-q) \tag{4.5}$$

$$\sum_{k=0}^{n} k^2 H_{2k+1} = (n^2+2)H_{2n+2} - (2n+1)H_{2n+1} - (p+2q) \tag{4.6}$$

$$\sum_{k=0}^{n} \sum_{j=0}^{k} H_j = H_{n+4} - (n+3)p - q \tag{4.7}$$

$$\sum_{k=0}^{n} k^2 H_k = (n^2+2)H_{n+2} - (2n-3)H_{n+3} - H_6 \tag{4.8}$$

$$\sum_{k=0}^{n} k^3 H_k = (n^3+6n-12)H_{n+2} - (3n^2-9n+19)H_{n+3} + (50p+31q) \tag{4.9}$$

$$\sum_{k=0}^{n} k^4 H_k = (n^4+12n^2-48n+98)H_{n+2} + (4n^3-18n^2+76n-159)H_{n+3} - (416p+257q) \tag{4.10}$$

$$5\sum_{k=0}^{n} (-1)^k H_{2k} = (-1)^n (H_{2n+2} + H_{2n}) - (p-3q) \tag{4.11}$$

$$5\sum_{k=0}^{n} (-1)^k H_{2k+1} = (-1)^n (H_{2n+3} + H_{2n+1}) + (2p-q) \tag{4.12}$$

$$5\sum_{k=0}^{n} (-1)^k kH_{2k} = (-1)^n (nH_{2n+2} + (n+1)H_{2n}) - q \tag{4.13}$$

$$(4.14)\qquad 5\sum_{k=0}^{n}(-1)^k kH_{2k+1} = (-1)^n(nH_{2n+3}+(n+1)H_{2n+1})-p$$

$$(4.15)\qquad 4\sum_{k=0}^{n}(-1)^k kH_{m+3k} = 2(-1)^n(n+1)H_{m+3n+1}-(-1)^nH_{m+3n+2}-H_{m-1}\quad (m=2,3,\cdots)$$

and so on.

REFERENCES

1. L. Carlitz and H. H. Ferns, "Some Fibonacci and Lucas Identities," *The Fibonacci Quarterly,* Vol. 8, No. 1 (Feb. 1970), pp. 61-73.
2. L. E. Dickson, *History of the Theory of Numbers,* Vol. 1, New York, 1952, pp. 393-407.
3. V. C. Harris, "Identities Involving Fibonacci Numbers," *The Fibonacci Quarterly,* Vol. 3, No. 3 (Oct. 1965), pp. 214–218.
4. M.R. Iyer, "Identities Involving Generalized Fibonacci Numbers," *The Fibonacci Quarterly,* Vol. 7, No. 1 (Feb. 1969), pp. 66–73.
5. V. E. Hoggatt, Jr., and M. Bicknell, "Fourth-Power Fibonacci Identities from Pascal's Triangle," *The Fibonacci Quarterly,* Vol. 2, No. 4 (Dec. 1964), pp. 261–266.
6. A.F. Horadam, "A Generalized Fibonacci Sequence, *Amer. Math. Monthly,* Vol. 68, No. 5, 1961, pp. 455-459.
7. A. F. Horadam, "Basic Properties of a Certain Generalized Sequence of Numbers," *The Fibonacci Quarterly,* Vol. 3, No. 3 (Oct. 1965), pp. 161-176.
8. K. Subba Rao, "Some Properties of Fibonacci Numbers," *Amer. Math. Monthly,* Vol. 60, No. 10, 1953, pp. 680-684.
9. J. E. Walton and A. F. Horadam, "Some Aspects of Generalized Fibonacci Numbers," *The Fibonacci Quarterly,* Vol. 12, No. 3 (Oct. 1974), pp. 241–250.
10. D. Zeitlin, "On Identities for Fibonacci Numbers," *Amer. Math. Monthly,* Vol. 70, No. 11, 1963, pp. 987–991.
11. D. Zeitlin, "Power Identities for Sequences Defined by $W_{n+2} = dW_{n+1} - cW_n$," *The Fibonacci Quarterly,* Vol. 3, No. 4 (Dec. 1965), pp. 241–256.

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[Continued from Page 271.]

where X is the largest root of

$$(3)\qquad x^4-x^3-3x^2+x+1 = 0.$$

The astonishing appearance of (1) stems from a peculiarity of (3). The Galois group of this quartic is the octic group (the symmetries of a square), and its resolvent cubic is therefore reducible:

$$(4)\qquad z^3-8z-7 = (z+1)(z^2-z-7) = 0.$$

The common discriminant of (3) and (4) equals $725 = 5^2\cdot 29$. While the quartic field $Q(X)$ contains $Q(\sqrt{5})$ as a subfield it does not contain $Q(\sqrt{29})$. Yet X can be computed from any root of (4). The rational root $z = -1$ gives $X = (A+1)/4$ while $z = (1+\sqrt{29})/2$ gives $X = (B+1)/4$.

It is clear that we can construct any number of such incredible identities from other quartics having an octic group. For example

$$x^4-x^3-5x^2-x+1 = 0$$

has the discriminant $4205 = 29^2\cdot 5$, and so the two expressions involve $\sqrt{5}$ and $\sqrt{29}$ once again. But this time $Q(\sqrt{29})$ is in $Q(X)$ and $Q(\sqrt{5})$ is not.

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