ADVANCED PROBLEMS AND SOLUTIONS

Edited By
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Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-239 Proposed by D. Finkel, Brooklyn, New York.

If a Fermat number $2^{2^n}+1$ is a product of precisely two primes, then it is well known that each prime is of the form $4m+1$ and each has a unique expression as the sum of two integer squares. Let the smaller prime be a^2+b^2, $a>b$; and the larger prime be c^2+d^2, $c>d$. Prove that

$$\left|\frac{c}{d}-\frac{d}{b}\right| \leq \frac{1}{100}.$$

Also, given that

$$2^{2^6}+1 = (274,177)(67,280,421,310,721),$$

and that

$$274,177 = 516^2+89^2,$$

express the 14-digit prime as a sum of two squares.

H-240 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Let

$$S(m,n,p) = (q)_n(q)_p \sum_{i=0}^{\min(n,p)} \frac{q^{mi+(n-i)(p-i)}}{(q)_i(q)_{n-i}(q)_{p-i}},$$

where

$$(q)_j = (1-q)(1-q^2)\cdots(1-q^j), \qquad (q)_o = 1.$$

Show that $S(m,n,p)$ is symmetric in m,n,p.

H-241 Proposed by R. Garfield, College of Insurance, New York, New York.

Prove that

$$\frac{1}{1-x^n} = \frac{1}{n}\sum_{k=0}^{n-1} \frac{1}{1-xe^{\frac{2k\pi}{n}i}}.$$

SOLUTIONS

GEE!

H-207 Proposed by C. Bridger, Springfield, Illinois

Define $G_k(x)$ by the relation

$$\frac{1}{1-(x^2+1)s^2-xs^3} = \sum_{n=0}^{\infty} G_k(x)s^k ,$$

where x is independent of s. (1) Find a recurrence formula connecting the $G_k(x)$. (2) Put $x = 1$ and find $G_k(1)$ in terms of Fibonacci numbers. (3) Also with $x = 1$, show that the sum of any four consecutive G numbers is a Lucas number.

Solution by the Proposer.

After carrying out the indicated division, we find

$$G_0(x) = 1,\ G_1(x) = 0,\ G_2(x) = x^2+1,\ G_3(x) = x,\ G_4(x) = (x^2+1)^2,$$

etc.

(1) Assume the recursion formula of the type

$$G_{k+3}(x) = pG_{k+2}(x) + qG_{k+1}(x) + rG_k(x),$$

and put $k = 0$, $k = 1$, and $k = 2$. The solution of the resulting equations gives $p = 0$, $q = x^2 + 1$, and $r = x$. So the recursion formula is

$$G_{k+3}(x) = (x^2+1)G_{k+1}(x) + xG_k(x).$$

(2) Put $x = 1$ to obtain

$$G_{k+3} = 2G_{k+1} + G_k .$$

This has the characteristic equation $z^3 - 2z - 1 = 0$, whose roots are

$$a = \frac{1+\sqrt{5}}{2},\quad b = \frac{1-\sqrt{5}}{2},\quad c = -1 .$$

Now,

$$\frac{a^k - b^k}{a-b} = F_k, \qquad \text{so} \qquad G_k(1) = F_k + (-1)^k .$$

(3) Use $F_{k+1} + F_{k-1} = L_k$ and $F_{k+2} + F_k = L_{k+1}$, replace F by G and add to obtain

$$G_{k+2} + G_{k+1} + G_k + G_{k-1} = L_{k+2} .$$

Also solved by G. Wulczyn, P. Tracy, P. Bruckman.

BOUNDS FOR A SUM

H-208 Proposed by P. Erdös, Budapest, Hungary.

Assume

$$\frac{n!}{a_1!a_2!\cdots a_k!} \qquad (a_i \geq 2,\ 1 \leq i \leq k),$$

is an integer. Show that the

$$\max \sum_{i=1}^{k} a_i < \frac{5}{2} n ,$$

where the maximum is to be taken with respect to all choices of the a_i's and k.

Solution by O.P. Lossers, Technological University Eindhoven, the Netherlands.

From the well known fact that the number $c_p(m)$ of prime factors p in $m!$ equals

$$c_p(m) = \left[\frac{m}{p}\right] + \left[\frac{m}{p^2}\right] + \left[\frac{m}{p^3}\right] + \cdots$$

($[x]$ denotes greatest integer in x), it follows that

$$c_2(2) = c_2(3) = 1,\ \frac{m}{2} \leq c_2(m) < m \quad (m \geq 4) \qquad c_3(m) < \frac{1}{2}m \quad (m \geq 2).$$

Now writing

$$a_1!a_2!\cdots a_k! = (2!)^{\alpha}(3!)^{\beta}b_1!\cdots b_{\ell}!\ ,$$

where $b_i \geqslant 4$ $(i = 1, \cdots, \ell)$ lower bounds for the number of factors 2 and 3 in $a_1!\cdots a_k!$ and *a fortiori* for $c_2(n)$ and $c_3(n)$ are found to be $\alpha+\beta+\tfrac{1}{2}\Sigma b_i$ and β, respectively. So

$$\Sigma a_j = 2\alpha+3\beta+\Sigma b_i \leqslant 2c_2(n)+c_3(n) < 2n+\frac{1}{2}n = \frac{5}{2}n\ .$$

Also solved by V. E. Hoggatt, Jr.

SEARCH!

H-209 (Corrected). Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Put

$$u_n = \frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}\ ,$$

where $\alpha+\beta = \alpha\beta = z$. determine the coefficients $C(n,k)$ such that

$$Z^n = \sum_{k=1}^{n} C(n,k)U_{n-k+1} \qquad (n \geqslant 1).$$

Solution by the Proposer.

It is easily verified that

$$\begin{aligned} z &= u_1 \\ z^2 &= u_2+u_1 \\ z^3 &= u_3+2u_2+2u_1 \\ z^4 &= u_4+3u_3+5u_2+5u_1 \\ z^5 &= u_5+4u_4+5u_3+14u_2+14u_1\ . \end{aligned}$$

Put

$$z^n = \sum_{k=1}^{n} C(n,k)u_{n-k+1}\ .$$

Since

$$(\alpha+\beta)u_k = (\alpha+\beta)\frac{\alpha^{k+1}-\beta^{k+1}}{\alpha-\beta} = \frac{\alpha^{k+2}-\beta^{k+2}}{\alpha-\beta}+\alpha\beta\frac{\alpha^k-\beta^k}{\alpha-\beta} = u_{k+1}+(\alpha+\beta)u_{k-1}\ ,$$

it follows that

$$(\alpha+\beta)u_k = u_1+u_2+\cdots+u_{k+1}\ .$$

Hence

$$z^{n+1} = \sum_{k=1}^{n} C(n,k)(\alpha+\beta)u_{n-k+1} = \sum_{k=1}^{n} C(n,k)\sum_{j=1}^{n-k+2} u_j = \sum_{j=1}^{n+1} u_j \sum_{k=1}^{n-j+2} C(n,k) = \sum_{j=1}^{n+1} u_{n-j+2}\sum_{k=1}^{j} C(n,k)\ .$$

It follows that $C(n,k)$ satisfies the recurrence

$$C(n+1,k) = \sum_{j=1}^{k} C(n,k)\ .$$

The first few values are easily computed $(1 \leqslant k \leqslant n \leqslant 5)$.

$$\begin{matrix} 1 \\ 1 & 1 \\ 1 & 2 & 2 \\ 1 & 3 & 5 & 5 \\ 1 & 4 & 9 & 14 & 14\ . \end{matrix}$$

Thus $C(n,k)$ can be identified with the number of sequences of positive integers $(a_1, a_2, \cdots, a_n)$ such that

$$\begin{cases} a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n \\ \quad a_i \leqslant i \qquad (i = 1, 2, \cdots, n) . \end{cases}$$

It is known (see for example L. Carlitz and J. Riordan, "Two Element Lattice Permutation Numbers and Their q-Generalization," *Duke Math. Journal,* Vol. 31 (1964), pp. 371–388) that

$$C(n,k) = \binom{n+k-2}{k-1} - \binom{n+k-2}{k-2} .$$

LUCAS CONDITION

H-210 Proposed by G. Wulczyn, Bucknell University, Lewisburg, Pennsylvania.

Show that a positive integer n is a Lucas number if and only if $5n^2 + 20$ or $5n^2 - 20$ is a square.

Solution by the Proposer.

I. (a) Let $n = L_{2m+1}$

$$5n^2 + 20 = 5(\alpha^{2m+1} + \beta^{2m+1})^2 + 20 = 5[\alpha^{4m+2} - 2(\alpha\beta)^{2m+1} + \beta^{4m+2}] = 25F_{2m+1}^2 .$$

(b) Let $n = L_{2m}$

$$5n^2 - 20 = 5(\alpha^{2m} + \beta^{2m})^2 - 20 = 5[\alpha^{4m} - 2(\alpha\beta)^{2m} + \beta^{4m}] = 25F_{2m}^2 .$$

II. $s^2 = 5n^2 + 20.$

(a) One solution chain is given by the rational part (for s) and the irrational part (for n) of

$$(5 + \sqrt{5})(9 + 4\sqrt{5})^t, \qquad t = 0, 1, 2, \cdots$$

with the irrational part also identical to L_{6t+1}. Let

$$(5 + \sqrt{5})(9 + 4\sqrt{5})^t = s_t + L_{6t+1}\sqrt{5}, \qquad s_t^2 = 5L_{6t+1}^2 + 20.$$

$$(5 + \sqrt{5})(9 + 4\sqrt{5})^{t+1} = 9s_t + 20L_{6t+1} + \sqrt{5}(9L_{6t+1} + 4s_t) .$$

$$9L_{6t+1} + 4\sqrt{5L_{6t+1}^2 + 20} = 9L_{6t+1} + 4\sqrt{5(\alpha^{6t+1} - \beta^{6t+1})^2} = 9L_{6t+1} + 20F_{6t+1}$$

$$\begin{aligned} L_{6t+7} = \alpha^{6t+7} + \beta^{6t+7} &= \alpha^{6t+1}(9 + 4\sqrt{5}) + \beta^{6t+1}(9 - 4\sqrt{5}) \\ &= 9L_{6t+1} + 20F_{6t+1} , \end{aligned}$$

(b) A second solution chain is given by the rational part (for s) and the irrational part (for n) of

$$(10 + 4\sqrt{5})(9 + 4\sqrt{5})^t, \qquad t = 0, 1, 2, \cdots .$$

The proof that the rational part of

$$(10 + 4\sqrt{5})(9 + 4\sqrt{5})^t$$

is identically L_{6t+3} is similar to that used in II (a).

(c) A third solution chain is given by the rational part (for s) and the irrational part (for n) of

$$(25 + 11\sqrt{5})(9 + 4\sqrt{5})^t, \qquad t = 0, 1, 2, \cdots .$$

The proof that the irrational part of

$$(25 + 11\sqrt{5})(9 + 4\sqrt{5})^t$$

is identically L_{6t+5} is similar to that used in II (a).

III. $s^2 = 5n^2 - 20.$

(a) One solution chain is given by the rational part (for s) and the irrational part (for n) of

$$(5 + 3\sqrt{5})(9 + 4\sqrt{5})^t, \qquad t = 0, 1, 2, \cdots .$$

Assume

$$(5+3\sqrt{5})(9+4\sqrt{5})^t = s_t + L_{6t+2}5, \qquad s_t^2 = 5L_{6t+2}^2 - 20.$$

$$(5+3\sqrt{5})(9+4\sqrt{5})^{t+1} = 9s_t + 20L_{6t+20} + \sqrt{5}(9L_{6t+2}+4s_t)$$

$$9L_{6t+2}+4s_t = 9L_{6t+2}+4\sqrt{5[(\alpha^{6t+2}+\beta^{6t+2})^2-4]} = 9L_{6t+2}+20F_{6t+2}$$

$$L_{6t+8} = \alpha^{6t+8}+\beta^{6t+8} = (9+4\sqrt{5})\alpha^{6t+2}+(9-4\sqrt{5})\beta^{6t+2}$$

$$= 9L_{6t+2}+20F_{6t+2} \quad .$$

(b) A second solution chain is given by the rational part (for s) and the irrational part (for n) of

$$(15+7\sqrt{5})(9+4\sqrt{5})^t, \qquad t = 0, 1, 2, \cdots .$$

The proof that the irrational part of

$$(15+7\sqrt{5})(9+4\sqrt{5})^t$$

is identical to L_{6t+4} is similar to that used in III (a).

(c) A third solution chain is given by the rational part (for s) and the irrational part (for n) of

$$(40+18\sqrt{5})(9+4\sqrt{5})^t, \qquad t = 0, 1, 2, \cdots .$$

The proof that the irrational part of

$$(40+18\sqrt{5})(9+4\sqrt{5})^t$$

is identical to L_{6t} is similar to that used in III (a).

Also solved by P. Bruckman, P. Tracy, and J. Ivie.

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$$y+1 \leqslant z < y+(x/n)$$

is a necessary condition for a solution. Thus, we see that there can be no solution for integer x, $1 \leqslant x \leqslant n$, a well known result (see [1, p. 744]). Again, if $y = n$, there is no solution for $1 \leqslant x \leqslant n$, a well known result (see [1, p. 744]). Our proof can also be used to establish the following general result.

Theorem 2. For $n \geqslant m \geqslant 2$ and integers $A \geqslant 1$, $B \geqslant 1$, the equation

$$Ax^n + By^m = Bz^m$$

has no solution whenever $Ax^{n-m+1} + Bmy \leqslant Bmz$.

REMARK. Theorem 2 gives Theorem 1 for $A = B$ and $n = m$.

REFERENCE

1. L.E. Dickson, *History of the Theory of Numbers,* Vol. 2, Diophantine Analysis, Carnegie Institute of Washington, 1919, Reprint by Chelsea, 1952.

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