A q-IDENTITY

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1. The object of this note is to prove the following q-identity:

$$(*)\qquad \sum_{k=0}^{n}(-1)^{n-k}\frac{(q)_n}{(q)_k}(a)_k(b)_k q^{\frac12(n-k)(n+k-1)}(ab)^{n-k} = (a)_{n+1}\sum_{k=0}^{n}(-1)^k\begin{bmatrix} n\\ k\end{bmatrix}q^{\frac12 k(k-1)}\frac{b^k}{1-q^{n-k}a}$$

$$= (b)_{n+1}\sum_{k=0}^{n}(-1)^k\begin{bmatrix} n\\ k\end{bmatrix}q^{\frac12 k(k-1)}\frac{a^k}{1-q^{n-k}b},$$

where

$$(a)_k = (a,q)_k = (1-a)(1-qa)\cdots(1-q^{k-1}a),\qquad (a)_0 = 1,$$

$$(q)_k = (q,q)_k = (1-q)(1-q^2)\cdots(1-q^k),\qquad (q)_0 = 1,$$

$$\begin{bmatrix} n\\ k\end{bmatrix} = \frac{(q)_n}{(q)_k(q)_{n-k}} = \begin{bmatrix} n\\ n-k\end{bmatrix}\qquad (0\le k\le n)$$

and q is not a t^{th} root of unity, $1 \le t \le n$.

Since each side of

$$(1)\qquad \sum_{k=0}^{n}(-1)^{n-k}\frac{(q)_n}{(q)_k}(a)_k(b)_k q^{\frac12(n-k)(n+k-1)}(ab)^{n-k} = (b)_{n+1}\sum_{k=0}^{n}(-1)^k\begin{bmatrix} n\\ k\end{bmatrix}q^{\frac12 k(k-1)}\frac{a^k}{1-q^{n-k}b}$$

is a polynomial in b of degree $\le n$, it will suffice to show that (1) holds for $b = q^{-r}$, $0 \le r \le n$.

We have

$$\left.\frac{(b)_{n+1}}{1-q^r b}\right|_{b=q^{-r}} = (1-q^{-r})\cdots(1-q^{-1})(1-q)(1-q^2)\cdots(1-q^{n-r}) = (-1)^r q^{-\frac12 r(r+1)}(q)_r(q)_{n-r}.$$

Thus the right-hand side of (1) reduces to

$$(2)\qquad (-1)^{n-r}\begin{bmatrix} n\\ n-r\end{bmatrix}q^{\frac12(n-r)(n-r-1)}(-1)^r q^{-\frac12 r(r+1)}(q)_r(q)_{n-r}a^{n-r} = (-1)^n(q)_n q^{\frac12 n(n-1)-nr}a^{n-r}.$$

As for the left-hand side, since

$$(q^{-r})_k = (1-q^{-r})(1-q^{-r+1})\cdots(1-q^{-r+k-1}) = (-1)^k q^{-rk+\frac12 k(k-1)}(1-q^r)(1-q^{r-1})\cdots(1-q^{r-k+1})$$

$$= \begin{cases}(-1)^k q^{-rk+\frac12 k(k-1)}(q)_r/(q)_{r-k} & (0\le k\le r)\\ 0 & (k>r)\end{cases},$$

we get

$$\sum_{k=0}^{r}(-1)^{n-k}\frac{(q)_n}{(q)_k}a^{n-k}q^{\frac12(n-k)(n+k-1)}\cdot(-1)^k q^{-rk+\frac12 k(k-1)}\frac{(q)_r}{(q)_{r-k}}q^{-r(n-k)} = (-1)^n(q)_n q^{\frac12 n(n-1)-nr}a^{n-r}\sum_{k=0}^{r}\begin{bmatrix} r\\ k\end{bmatrix}(a)_k a$$

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We shall now show that

$$\sum_{k=0}^{r} \begin{bmatrix} r \\ k \end{bmatrix} (a)_k a^{r-k} = 1 \qquad (r = 0, 1, 2, \cdots), \tag{3}$$

so that the left-hand side of (1) is equal to

$$(-1)^n (q)_n q^{\frac{1}{2}n(n-1)-nr} a^{n-r}$$

in agreement with (2).

To prove (3) we take

$$\sum_{r=0}^{\infty} \frac{x^r}{(q)_r} \sum_{k=0}^{r} \begin{bmatrix} r \\ k \end{bmatrix} (a)_k a^{r-k} = \sum_{k=0}^{\infty} \frac{(a)_k}{(q)_k} x^k \sum_{r=0}^{\infty} \frac{a^r x^r}{(q)_r}.$$

By a well known identity

$$\sum_{k=0}^{\infty} \frac{(a)_k}{(q)_k} x^k = \frac{e(x)}{e(ax)},$$

where

$$e(x) = \sum_{n=0}^{\infty} \frac{x^n}{(q)_n} = \prod_{n=0}^{\infty} (1 - q^n x)^{-1}. \tag{4}$$

Thus

$$\sum_{r=0}^{\infty} \frac{x^r}{(q)_r} \sum_{k=0}^{r} \begin{bmatrix} r \\ k \end{bmatrix} (a)_k a^{r-k} = \frac{e(x)}{e(ax)} e(ax) = e(x)$$

and (3) follows at once.

This evidently completes the proof of (*).

2. The identity (*) can also be proved by making use of the q-analog of Gauss's theorem (see for example [1, p. 68]):

$$\sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(q)_k (x)_k} \left(\frac{x}{ab} \right)^k = \frac{e(x) e(x/ab)}{e(x/a) e(x/b)}, \tag{5}$$

where $e(x)$ is defined by (4).

Define the operator E by means of

$$E^n f(x) = f(q^n x) \qquad (n = 0, 1, 2, \cdots)$$

and Δ^n by means of the operational formula

$$\Delta^n = (1 - E)(q - E) \cdots (q^{n-1} - E).$$

Then it is easily verified that

$$\Delta^n = \sum_{r=0}^{n} (-1)^{n-r} \begin{bmatrix} n \\ r \end{bmatrix} q^{\frac{1}{2}r(r-1)} E^{n-r}.$$

It follows that

$$\Delta^n x^k = \sum_{r=0}^{n} (-1)^{n-r} \begin{bmatrix} n \\ r \end{bmatrix} q^{\frac{1}{2}r(r-1)} q^{(n-r)k} x^k = (q^k - 1)(q^k - q) \cdots (q^k - q^{n-1}) x^k,$$

so that

$$\Delta^n x^k = \begin{cases} 0 & (n > k) \\ (-1)^k q^{\frac{1}{2}k(k-1)} (q)_k x^k & (n = k) \end{cases} \tag{6}$$

Now multiply both sides of (5) by $(x)_n$ and apply Δ^n. Then divide by x^n and put $x = 0$. In view of (6) the LHS becomes

$$\sum_{k=0}^{n} \frac{(a)_k(b)_k}{(q)_k}(ab)^{-k} \cdot (-1)^{n-k} q^{\frac{1}{2}(n-k)(n-k-1)+k(n-k)} \cdot (-1)^n q^{\frac{1}{2}n(n-1)}(q)_n$$

(7)
$$= (-1)^n q^{\frac{1}{2}n(n-1)} \sum_{k=0}^{n} (-1)^{n-k} \frac{(q)_n}{(q)_k}(a)_k(b)_k q^{\frac{1}{2}(n-k)(n+k-1)}(ab)^{-k}\ .$$

As for the RHS, we have first

$$(x)_n \frac{e(x)e(x/ab)}{e(x/a)e(x/b)} = \frac{e(q^n x)e(x/ab)}{e(x/a)e(x/b)}$$
$$= \sum_{j=0}^{\infty} \frac{(q^{-n}/a)_j}{(q)_j}(q^n x)^j \sum_{k=0}^{\infty} \frac{(a)_k}{(q)_k}\left(\frac{x}{ab}\right)^k$$
$$= \sum_{r=0}^{\infty} \frac{x^r}{(q)_r} \sum_{k=0}^{r} \begin{bmatrix} r \\ k \end{bmatrix} (a)_k(ab)^{-k}(q^{-n}/a)_{r-k} q^{n(r-k)}\ .$$

Apply Δ^n, divide by x^n and put $x = 0$. We get

(8)
$$(-1)^n q^{\frac{1}{2}n(n-1)} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (a)_k(ab)^{-k}(q^{-n}/a)_{n-k} q^{n(n-k)}\ .$$

Since

$$(q^{-n}/a)_{n-k} = (1-q^{-n}/a)(1-q^{-n+1}/a)\cdots(1-q^{-k-1}/a) = (-1)^{n-k}a^{-n+k}q^{-\frac{1}{2}n(n+1)+\frac{1}{2}k(k+1)}$$
$$\cdot (1-q^{k+1}a)(1-q^{k+2}a)\cdots(1-q^n a)\ ,$$

(8) becomes

$$q^{\frac{1}{2}n(n-1)}(ab)^{-n} \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\frac{1}{2}n(n-1)-nk+\frac{1}{2}k(k+1)} b^{n-k} \frac{(a)_{n+1}}{1-q^k a}$$
$$= (-1)^n q^{\frac{1}{2}n(n-1)}(ab)^{-n} \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\frac{1}{2}k(k-1)} b^k \frac{(a)_{n+1}}{1-q^{n-k}a}\ .$$

Comparing this with (7) it is clear that we have proved (*).

3. We have

$$\sum_{n=0}^{\infty} \frac{x^n}{(q)_n} \sum_{k=0}^{n} (-1)^{n-k} \frac{(q)_n}{(q)_k}(a)_k(b)_k q^{\frac{1}{2}(n-k)(n+k-1)}(ab)^{n-k}$$
$$= \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(q)_k} x^k \sum_{n=0}^{\infty} (-1)^n q^{\frac{1}{2}n(n-1)}(q^k abx)^n\ .$$

Also, since

$$(a)_{n+1} = (a)_{n-k}(1-q^{n-k}a)(q^{n-k+1}a)_k\ ,$$

$$\sum_{n=0}^{\infty} \frac{(a)_{n+1}}{(q)_n} x^n \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\frac{1}{2}k(k-1)} \frac{b^k}{1-q^{n-k}a} = \sum_{k=0}^{\infty} (-1)^k q^{\frac{1}{2}k(k-1)} \frac{b^k x^k}{(q)_k} \sum_{n=0}^{\infty} \frac{(a)_n}{(q)_n}(q^{n+1}a)_k x^n$$
$$= \sum_{n=0}^{\infty} \frac{(a)_n}{(q)_n} x^n \sum_{k=0}^{\infty} (-1)^k q^{\frac{1}{2}k(k-1)} \frac{(q^{n+1}a)_k}{(q)_k}(bx)^k\ .$$

Thus (*) is equivalent to the identity

$$\sum_{n=0}^{\infty}(-1)^n q^{\frac{1}{2}n(n-1)}(abx)^n\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(q)_k}(q^nx)^k = \sum_{n=0}^{\infty}\frac{(a)_n}{(q)_n}x^n\sum_{k=0}^{\infty}(-1)^k q^{\frac{1}{2}k(k-1)}\frac{(q^{n+1}a)_k}{(q)_k}(bx)^k$$

$$(9)\qquad = \sum_{n=0}^{\infty}\frac{(b)_n}{(q)_n}x^n\sum_{k=0}^{\infty}(-1)^k q^{\frac{1}{2}k(k-1)}\frac{(q^{n+1}b)_k}{(q)_k}(ax)^k ,$$

where now $|q| < 1$.

4. The following special cases of (*) may be noted. For $b = q$ we have

$$\sum_{k=0}^{n}(-1)^{n-k}(a)_k q^{\frac{1}{2}k(n-k)(n+k+1)}a^{n-k} = \frac{(a)_{n+1}}{(q)_n}\sum_{k=0}^{n}(-1)^k\begin{bmatrix} n \\ k\end{bmatrix}\frac{q^{\frac{1}{2}k(k+1)}}{1-q^{n-k}a}$$

$$(10)\qquad = (1-q^{n+1})\sum_{k=0}^{n}(-1)^k\begin{bmatrix} n \\ k\end{bmatrix}q^{\frac{1}{2}k(k-1)}\frac{a^k}{1-q^{n-k+1}} .$$

For $a = q$ this reduces to

$$(11)\qquad \sum_{k=0}^{n}(-1)^{n-k}(q)_k q^{\frac{1}{2}(n-k)(n+k+3)} = (1-q^{n+1})\sum_{k=0}^{n}(-1)^k\begin{bmatrix} n \\ k\end{bmatrix}\frac{q^{\frac{1}{2}k(k+1)}}{1-q^{n-k+1}} .$$

Since

$$\begin{bmatrix} n+1 \\ k\end{bmatrix} = \frac{1-q^{n+1}}{1-q^{n-k+1}}\begin{bmatrix} n \\ k\end{bmatrix}$$

and

$$\sum_{k=0}^{n}(-1)^k\begin{bmatrix} n+1 \\ k\end{bmatrix}q^{\frac{1}{2}k(k+1)} = (q)_{n+1} + (-1)^n q^{\frac{1}{2}(n+1)(n+2)} ,$$

(11) becomes

$$(12)\qquad \sum_{k=0}^{n}(-1)^{n-k}(q)_k q^{\frac{1}{2}(n-k)(n+k+3)} = (q)_{n+1} + (-1)^n q^{\frac{1}{2}(n+1)(n+2)} .$$

Somewhat more generally, it follows from (10) that

$$(13)\qquad \sum_{k=0}^{n}(-1)^{n-k}(a)_k q^{\frac{1}{2}(n-k)(n+k+1)}a^{n-k} = (a)_{n+1} + (-1)^n q^{\frac{1}{2}n(n+1)}a^{n+1} .$$

We shall give a direct proof of (13). The formula evidently holds for $n = 0$. Assuming that it holds up to and including the value n, we replace a by qa and multiply both sides by $1-a$. Thus

$$\sum_{k=0}^{n}(-1)^{n-k}(a)_{k+1}q^{\frac{1}{2}(n-k)(n+k+3)}a^{n-k} = (a)_{n+2} + (-1)^n q^{\frac{1}{2}(n+1)(n+2)}a^{n+1}(1-a) .$$

Hence

$$\sum_{k=0}^{n+1}(-1)^{n-k+1}(a)_k q^{\frac{1}{2}(n-k+1)(n+k+2)}a^{n-k+1} = (a)_{n+2} + (-1)^n q^{\frac{1}{2}(n+1)(n+2)}a^{n+1}(1-a) + (-1)^{n+1}$$

$$\cdot\, q^{\frac{1}{2}(n+1)(n+2)}a^{n+1} = (a)_{n+2} + (-1)^{n+1}q^{\frac{1}{2}(n+1)(n+2)}a^{n+2} .$$

REFERENCE

1. W.N. Bailey, *Generalized Hypergeometric Series,* Cambridge, 1935.

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