THE EVALUATION OF CERTAIN ARITHMETIC SUMS

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1. In this paper we evaluate certain cases of the expression

$$\sum \max(A_1, A_2, \cdots, A_n), \tag{1.1}$$

where

$$A_k = a_{k1} + \cdots + a_{km_k}$$

and where the sum is over all the a's, each ranging from zero to some positive integer. We also consider analogous sums for min. For example we obtain, from some general results which we establish, the formula

$$\sum_{a,b,c,d=0}^{r} \max(a+b, c+d) = 22\binom{r}{1} + 170\binom{r}{2} + 420\binom{r}{3} + 420\binom{r}{4} + 148\binom{r}{5}.$$

Some general properties of more general cases of (1.1) are established.

Solutions of many problems, particularly combinatorial problems, are often expressed in terms of such sums. For example, without going into detail, we frequently encounter sums of max and min in problems of enumerating arrays. See H. Anand *et al.* [1] and Carlitz [2] for details.

In a related work, Carlitz [3] and [4] obtains, and relates to other problems, generating functions for $\max(n_1, \cdots, n_k)$ and $\min(n_1, \cdots, n_k)$. More generally, he evaluates

$$\sum_{n_1,\cdots,n_k=0}^{\infty} M_r(n_1, \cdots, n_k) x_1^{n_1} \cdots x_k^{n_k} \qquad (r = 1, \cdots, k),$$

where $M_r(n_1, \cdots, n_k)$ is defined by the following two properties: (a) it is symmetric in $n_1, \cdots, n_k$; (b) if $n_1 \leqslant n_2 \leqslant \cdots \leqslant n_k$ then

$$M_r(n_1, \cdots, n_k) = n_r \qquad (r = 1, \cdots, k).$$

He also evaluates the related series

$$\sum_{n_1,\cdots,n_k=0}^{\infty} x_1^{n_1} \cdots x_k^{n_k} z^{M_r(n_1,\cdots,n_k)} \qquad (r = 1, \cdots, k).$$

Roselle [6] examines the relationship between this series and the Eulerian function.

Other than [3], [4], [6] and this paper, there apparently has been very little published on problems of this nature.

A number of techniques are employed to solve various aspects of the problem and computer computation was necessary in some instances.

The main results of this paper are (3.3), (4.3), (4.7), (4.8), (4.11), (4.12), (4.13), (4.14), (4.15), (5.7), (5.9), (5.10), (5.13), (5.14), (5.15) and (5.16).

2. Preliminary to our discussion we need some basic properties of Eulerian polynomials. The n^{th} Eulerian polynomial, $a_n(x)$, is defined, following Riordan [5], by

$$a_n(x) = (1-x)^{n+1} \sum_{k=0}^{\infty} k^n x^k .$$

From this definition we get

$$a_n(x) = nxa_{n-1}(x) + x(1-x)Da_{n-1}(x) ,$$

where D is the differential operator. Hence, the first few polynomials, which we will use later, are

$$a_0(x) = 1, \qquad a_1(x) = x, \qquad a_2(x) = x^2 + x,$$
$$a_3(x) = x^3 + 4x^2 + x, \qquad a_4(x) = x^4 + 11x^3 + 11x^2 + x.$$

A recurrence and a table for the coefficients (Eulerian numbers) of Eulerian polynomials and a generating function for $a_n(x)$ may be found in [5; pp. 39, 215].

As for convenient notation we write

$$\genfrac{}{}{0pt}{}{max}{min}\,(a,b) ,$$

where we wish to discuss both $max\,(a,b)$ and $min\,(a,b)$.

Also we adopt the convention

$$\phi = \phi_n = (1-x_1)(1-x_2)\cdots(1-x_n).$$

3. Taking the simplest case first we evaluate the sum

$$\sum_{i_1,\cdots,i_n=0}^{r_1,\cdots,r_n} min\,(i_1,\cdots,i_n) .$$

To do this, we put

$$F(x_1,\cdots,x_n) = \sum_{r_1,\cdots,r_n=0}^{\infty} \sum_{i_1,\cdots,i_n=0}^{r_1,\cdots,r_n} min\,(i_1,\cdots,i_n)x_1^{r_1}\cdots x_n^{r_n} , \tag{3.1}$$

which becomes

$$F(x_1,\cdots,x_n) = \phi^{-1} \sum_{i_1,\cdots,i_n=0}^{\infty} min\,(i_1,\cdots,i_n)x_1^{i_1}\cdots x_n^{i_n} .$$

Now since

$$min\,(i_1,\cdots,i_n) = \sum_{j=0}^{min(i_1-1,\cdots,i_n-1)} 1 = \sum_{\substack{k_1+j+1=i_1\\ \cdots \\ k_n+j+1=i_n}} 1,$$

it follows immediately that

$$\sum_{i_1,\cdots,i_n=0}^{\infty} min\,(i_1,\cdots,i_n)x_1^{i_1}\cdots x_n^{i_n} = \phi^{-1}\frac{x_1x_2\cdots x_n}{(1-x_1x_2\cdots x_n)} .$$

Therefore,

$$F(x_1,\cdots,x_n) = \phi^{-2}\frac{x_1x_2\cdots x_n}{1-x_1x_2\cdots x_n} . \tag{3.2}$$

Comparing the coefficients in the expansion of (3.2) with those of (3.1) gives

$$\sum_{i_1,\cdots,i_n=0}^{r_1,\cdots,r_n} min\,(i_1,\cdots,i_n) = \sum_{j\leqslant min(r_1,\cdots,r_n)} \prod_{i=1}^{n} (r_i-j) . \tag{3.3}$$

If $r_1 = \cdots = r_n = r$ then the right side of (3.3) reduces to the familiar series

$$\sum_{j=0}^{r} j^n .$$

4. The sum

$$\sum_{i_1,\cdots,i_n=0}^{r_1,\cdots,r_n} \max(i_1,\cdots,i_n) \tag{4.1}$$

is apparently more difficult to evaluate. If $n = 2$ we may use (3.3) and the identity

$$\min(i,j) + \max(i,j) = i + j \tag{4.2}$$

to get

$$\sum_{i,j=0}^{r,s} \max(i,j) = \tfrac{1}{2}r(r+1)(s+1) + s(s+1)(r+1) - \sum_{j=0}^{\min(r,s)} (r-j)(s-j)\,. \tag{4.3}$$

Now, considering the case of (4.1) where all the r's are equal we define

$$F_n(r) = \sum_{i_1,\cdots,i_n=0}^{r} \max(i_1,\cdots,i_n)\,. \tag{4.4}$$

Then by an inclusion-exclusion argument,

$$F_n(r) = \binom{n}{1}\sum_{i_1,\cdots,i_{n-1}=0}^{r} \max(i_1,\cdots,i_{n-1},r) - \binom{n}{2}\sum_{i_1,\cdots,i_{n-2}=0}^{r} \max(i_1,\cdots,i_{n-2},r,r)$$

$$+ \cdots + (-1)^{n+1}\binom{n}{n}\max(r,\cdots,r) + F_n(r-1)$$

or

$$F_n(r) = r\sum_{k=1}^{n}\binom{n}{k}(-1)^{k+1}(r+1)^{n-k} + F_n(r-1) \qquad (r \geq 1)\,. \tag{4.5}$$

The expression

$$\sum_{k=1}^{n}\binom{n}{k}(-1)^{k+1}(r+1)^{n-k} \tag{4.6}$$

may be simplified, according to the binomial theorem, giving

$$(r+1)^n - r^n$$

so that (4.5) becomes

$$F_n(r) = r(r+1)^n - r^{n+1} + F_n(r-1) \qquad (r \geq 1)\,. \tag{4.7}$$

Applying (4.7) to $F_n(r-1)$ we get

$$F_n(r) = r(r+1)^n - r^{n+1} + (r-1)r^n - (r-1)^{n+1} + F_n(r-2)$$

and continuing in this manner we eventually arrive at

$$F_n(r) = \sum_{k=0}^{r-1}(r-k)(r+1-k)^n - \sum_{k=0}^{r-1}(r-k)^{n+1}\,.$$

$$F_n(r) = \sum_{k=0}^{r} k(k+1)^n - \sum_{k=0}^{r} k^{n+1} = \sum_{k=0}^{r} k((k+1)^n - k^n) = \sum_{k=0}^{r} k\sum_{j=0}^{n-1}\binom{n}{j}k^j\,.$$

Now by comparing the last expression with (4.4) we have

$$F_n(r) = \sum_{i_1,\cdots,i_n=0}^{r} \max(i_1,\cdots,i_n) = \sum_{k=0}^{r}\sum_{j=0}^{n-1}\binom{n}{j}k^{j+1} \tag{4.8}$$

or, in the usual notation for Bernoulli polynomials,

$$\sum_{i_1,\cdots,i_n=0}^{r} \max(i_1,\cdots,i_n) = \sum_{j=0}^{n-1} \binom{n}{j} \frac{B_{j+2}(r+1)-B_{j+2}(0)}{j+2} . \tag{4.9}$$

The first few special cases of $F_n(r)$, obtained from (4.7) are

$$F_n(1) = 2^n - 1, \quad F_n(2) = 2\cdot 3^n - 2^n - 1, \quad \text{and} \quad F_n(3) = 3\cdot 4^n - 3^{n+1} + 2\cdot 3^n - 2^n - 1 .$$

Next we evaluate the generating function

$$\sum_{n=0}^{\infty} F_n(r)x^n . \tag{4.10}$$

From (4.8), (4.10) becomes

$$\sum_{n=0}^{\infty}\sum_{j=0}^{n-1}\binom{n}{j}\sum_{k=0}^{r} k^{j+1}x^n = \sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}\sum_{k=0}^{r} k^{j+1}x^n - \sum_{n=0}^{\infty}\binom{n}{n}\sum_{k=0}^{r} k^{n+1}x^n$$

$$= \sum_{k=0}^{r} k \sum_{n=0}^{\infty} x^n \sum_{j=0}^{\infty}\binom{n+j}{j}(kx)^j - \sum_{k=0}^{r} k \sum_{n=0}^{\infty}(kx)^n$$

$$= \sum_{k=0}^{r} k \sum_{n=0}^{\infty} x^n (1-kx)^{-n-1} - \sum_{k=0}^{r} k(1-kx)^{-1}$$

$$= \sum_{k=0}^{r} k(1-kx-x)^{-1} - \sum_{k=0}^{r} k(1-kx)^{-1}$$

$$= x\sum_{k=0}^{r} k(1-(k+1)x)^{-1}(1-kx)^{-1} .$$

We have therefore proved

$$\sum_{n=0}^{\infty} F_n(r)x^n = \sum_{n=0}^{\infty}\sum_{i_1,\cdots,i_n=0}^{r} \max(i_1,\cdots,i_n)x^n = x\sum_{k=0}^{r} k(1-(k+1)x)^{-1}(1-kx)^{-1} . \tag{4.11}$$

From (4.11) it is easy to see that

$$\sum_{n,r=0}^{\infty} F_n(r)x^ny^r = \frac{x}{(1-y)}\sum_{k=0}^{\infty}\frac{ky^k}{(1-(k+1)x)(1-kx)} .$$

If, on the other hand, we define

$$G_n(y) = \sum_{r=0}^{\infty} F_n(r)y^r$$

then using the recurrence (4.7) and setting $F_n(-1) = 0$ we have

$$G_n(y) = \sum_{r=0}^{\infty}(r(r+1)^n - r^{n+1})y^r + \sum_{r=0}^{\infty} F_n(r-1)y^r = \sum_{r=0}^{\infty}(r(r+1)^n - r^{n+1})y^r + y\sum_{r=0}^{\infty} F_n(r)y^r .$$

Then

$$(1-y)G_n(y) = \sum_{r=0}^{\infty} (r(r+1)^n - r^{n+1})y^r = \sum_{r=0}^{\infty} r \sum_{k=0}^{n} \binom{n}{k} r^k y^r - \sum_{r=0}^{\infty} r^{n+1} y^r$$

$$= \sum_{k=0}^{n} \binom{n}{k} \sum_{r=0}^{\infty} r^{k+1} y^r - \sum_{r=0}^{\infty} r^{n+1} y^r$$

$$= \sum_{k=0}^{n} \binom{n}{k} \frac{a_{k+1}(y)}{(1-y)^{k+2}} - \frac{a_{n+1}(y)}{(1-y)^{n+2}} = \sum_{k=0}^{n-1} \binom{n}{k} \frac{a_{k+1}(y)}{(1-y)^{k+2}} ,$$

where $a_p(x)$ is the p^{th} Eulerian polynomial introduced in Section 2.

Therefore, we have

$$G_n(y) = \sum_{r=0}^{\infty} \sum_{i_1,\cdots,i_n=0}^{r} \max(i_1,\cdots,i_n)\, y^r = \sum_{k=0}^{n-1} \binom{n}{k} \frac{a_{k+1}(y)}{(1-y)^{k+3}} . \tag{4.12}$$

From (4.12) and from the list of Eulerian polynomials in Section 2 we easily arrive at the special cases

$$G_2(y) = \frac{3y+y^2}{(1-y)^4}, \qquad G_3(y) = \frac{7y+10y^2+y^3}{(1-y)^5}, \quad \text{and} \quad G_4(y) = \frac{15y+55y^2+25y^3+y^4}{(1-y)^6} .$$

The expansions of each of these generating functions yield, for $r \geqslant 0$,

$$\sum_{a,b=0}^{r} \max(a,b) = 3\binom{r+2}{3} + \binom{r+1}{3} , \tag{4.13}$$

$$\sum_{a,b,c=0}^{r} \max(a,b,c) = 7\binom{r+3}{4} + 10\binom{r+2}{4} + \binom{r+1}{4} , \tag{4.14}$$

and

$$\sum_{a,b,c,d=0}^{r} \max(a,b,c,d) = 15\binom{r+4}{5} + 55\binom{r+3}{5} + 25\binom{r+2}{5} + \binom{r+1}{5} , \tag{4.15}$$

respectively.

In general, from (4.9) it may be seen that $F_n(r)$ is a polynomial in r of degree $n+1$.

5. In this section, we consider $A(r,m,n)$ and $B(r,m,n)$, where

$$A(r,m,n) = \sum_{a_1,\cdots,a_m,b_1,\cdots,b_n=0}^{r} \max(a_1+\cdots+a_m,\, b_1+\cdots+b_n) , \tag{5.1}$$

and

$$B(r,m,n) = \sum_{a_1,\cdots,a_m,b_1,\cdots,b_n=0}^{r} \min(a_1+\cdots+a_m,\, b_1+\cdots+b_n) . \tag{5.2}$$

It is convenient to let the expression $a,b=0$ mean $a_1,\cdots,a_m,b_1,\cdots,b_n = 0$.

Using the formula

$$\max_{\min}(a,b) = \tfrac{1}{2}(a+b \pm |a-b|) ,$$

(5.2) becomes

$$(5.3)\qquad B(r,m,n) = \tfrac{1}{2}\sum_{a,b=0}^{r}(a_1+\cdots+a_m+b_1+\cdots+b_n) - \tfrac{1}{2}\sum_{a,b=0}^{r}\left|a_1+\cdots+a_m-b_1-\cdots-b_n\right|$$

$$= \frac{(m+n)r(r+1)^{m+n}}{4} - \tfrac{1}{2}\sum_{a,b=0}^{r}\left|a_1+\cdots+a_m-b_1-\cdots-b_n\right| .$$

Now

$$(5.4)\qquad \sum_{a,b=0}^{r} a_1+\cdots+a_m-b_1-\cdots-b_n = -\sum_{I}(a_1+\cdots+a_m-b_1-\cdots-b_n)+\sum_{II}(a_1+\cdots+a_m-b_1-\cdots-b_n),$$

where I and II stand for

$$a_1+\cdots+a_m \leqslant b_1+\cdots+b_n \leqslant r\max(m,n)$$

and

$$b_1+\cdots+b_n \leqslant a_1+\cdots+a_m \leqslant r\max(m,n).$$

respectively, and where it is understood that $a_i, b_i \leqslant r$.

Moreover,

$$\sum_{I}(a_1+\cdots+a_m-b_1-\cdots-b_n) = \sum_{k=0}^{r\max(m,n)}\ \sum_{\substack{b_1+\cdots+b_n=k\\ b_i\leqslant r}}\ \sum_{\substack{a_1+\cdots+a_m\leqslant k\\ a_i\leqslant r}}(a_1+\cdots+a_m-k)$$

$$= \sum_{k=0}^{r\max(m,n)}\ \sum_{j=0}^{k}\ \sum_{\substack{b_1+\cdots+b_n=k\\ b_i\leqslant r}}\ \sum_{\substack{a_1+\cdots+a_m=j\\ a_i\leqslant r}}(j-k) .$$

Now let $P(a,b,c)$ be the number of partitions of $a>0$ into at most b parts, each part $\leqslant c$, and let $P(0,b,c)=1$.

$$(5.5)\qquad \sum_{I}(a_1+\cdots+a_m-b_1-\cdots-b_n) = \sum_{k=0}^{r\max(m,n)}\ \sum_{j=0}^{k}(k-j)P(k,n,r)P(j,m,r) .$$

By a similar argument we find

$$(5.6)\qquad \sum_{II}(a_1+\cdots+a_m-b_1-\cdots-b_n) = \sum_{k=0}^{r\max(m,n)}\ \sum_{j=0}^{k}(k-j)P(k,m,r)P(j,n,r) .$$

By substituting (5.5) and (5.6) into (5.4) and referring to (5.3), and then by applying this same argument for $A(r,m,n)$ we find that $A(r,m,n)$ and $B(r,m,n)$ are given by

$$(5.7)\qquad \sum_{a_1,\cdots,a_m,b_1,\cdots,b_n\leqslant r}\ {\max\atop\min}(a_1+\cdots+a_m,\,b_1+\cdots+b_n) = \frac{(m+n)r(r+1)^{m+n}}{4}$$

$$\pm\ \tfrac{1}{2}\sum_{k=0}^{r\max(m,n)}\ \sum_{j=0}^{k}(k-j)\left[P(k,n,r)P(j,m,r)+P(k,m,r)P(j,n,r)\right].$$

respectively.

Now, although (5.7) expresses our problem in terms of a difficult partition problem the formula is nevertheless very useful in that it affords us a method of determining the generating function. Here we observe, from (5.7), that $A(r,m,n)$ and $B(r,m,n)$ are polynomials in r. Furthermore, their degrees are less than or equal to $m+n+1$, for if not, then for large values of r, $B(r,m,n)$ will be negative. In fact, in view of special cases, we may conjecture that the degree of $A(r,m,n)$ and $B(r,m,n)$ is precisely $m+n+1$.

We may now evaluate several cases. First, we consider $B(r,2,1)$.

$$B(r,2,1) = \sum_{a,b,c=0}^{r} \min(a+b,c) = \sum_{\substack{a+b\leq c\\ a,b,c\leq r}} (a+b) + \sum_{\substack{a+b\geq c\\ a,b,c\leq r}} c - \sum_{c=0}^{r}\sum_{a+b=c} c\ .$$

After some manipulation it is seen that

$$\sum_{\substack{a+b\leq c\\ a,b,c\leq r}} (a+b) = \sum_{c=0}^{r}\sum_{k=0}^{c}\sum_{a+b=k} k = \frac{1}{3}\sum_{c=0}^{r}(c^3+3c^2+2c)$$

and

$$\sum_{c=0}^{r}\sum_{a+b=c} c = \sum_{c=0}^{r} c(c+1)\ .$$

Then we have

$$B(r,2,1) = \sum_{c=0}^{r}\left(\frac{1}{3}k^3 - \frac{1}{3}k\right) + \sum_{\substack{a+b\geq c\\ a,b,c\leq r}} c = \frac{1}{12}r^2(r+1)^2 - \frac{1}{6}r(r+1) + \sum_{a,b=0}^{r}\ \sum_{c=0}^{\min(a+b,r)} c.$$

Since the degree of $B(r,2,1)$ is at most $2+1+1$, it suffices to compute $B(r,2,1)$ for $r = 1, 2, 3, 4$. If we put

$$K_r = \sum_{a,b=0}^{r}\ \sum_{c=0}^{\min(a+b,r)} c$$

so that

(5.8) $$B(r,2,1) = \frac{1}{12}r^2(r+1)^2 - \frac{1}{6}r(r+1) + K_r$$

then it is easy to compute $K_1 = 3$, $K_2 = 20$, $K_3 = 71$ and $K_4 = 185$. Then from (5.8), we have

$$B(1,2,1) = 3,\qquad B(2,2,1) = 22,\qquad B(3,2,1) = 81 \quad\text{and}\quad B(4,2,1) = 215.$$

Then we find the differences

$$\Delta B(r,2,1) = 3,\qquad \Delta^2 B(r,2,1) = 16,\qquad \Delta^3 B(r,2,1) = 24,\qquad \Delta^4 B(r,2,1) = 11.$$

Substituting these values into Newton's expansion for the generating function we have

$$\sum_{k=0} B(r,2,1)x^k = \sum_{k=0}^{4} \Delta^k B(0,2,1)\frac{x^k}{(1-x)^{k+1}} = \frac{x}{(1-x)^5}(3+7x+x^2)\ .$$

Upon expanding we get

(5.9) $$B(r,2,1) = \sum_{a,b,c=0}^{r} \min(a+b,c) = 3\binom{r+3}{4} + 7\binom{r+2}{4} + \binom{r+1}{4}\ .$$

From (4.2),

$$\sum_{a,b,c=0}^{r} (\max(a+b,c) + \min(a+b,c)) = \sum_{a,b,c=0}^{r} (a+b+c)$$

which, from (5.9) reduces to

(5.10) $$\sum_{a,b,c=0}^{r} \max(a+b,c) = 3\binom{r+3}{4} + 7\binom{r+2}{4} + \binom{r+1}{4} + \frac{3}{2}r(r+1)^3\ .$$

Next we put $B(r,2,2) = B(r)$ for brevity, and consider

$$B(r) = \sum_{a,b,c=0}^{r} \min(a+b,\, c+d) \tag{5.11}$$

From (5.7) we have

$$B(r) = r(r+1)^4 - \sum_{k=0}^{2r} \sum_{j=0}^{k} (k-j)P(k,2,r)P(j,2,r)\,.$$

Hence $B(r)$ is a polynomial in r of degree $\leqslant 5$. Therefore the generating function is

$$\sum_{r=0}^{\infty} B(r)x^r = \sum_{i=0}^{5} \Delta^i B(0)\, \frac{x^i}{(1-x)^{i+1}}\;.$$

Resorting to a computer to calculate $\Delta^i B(0)$ for $i = 1, 2, 3, 4, 5,$ (5.12) becomes

$$\sum_{r=0}^{\infty} B(r)x = \frac{10x}{(1-x)^2} + \frac{90x^2}{(1-x)^3} + \frac{240x^3}{(1-x)^4} + \frac{252x^4}{(1-x)^5} + \frac{92x^5}{(1-x)^6}\;.$$

Expanding for the coefficients we find

$$\sum_{a,b,c,d=0}^{r} \min(a+b,\, c+d) = 10\binom{r}{1} + 90\binom{r}{2} + 240\binom{r}{3} + 252\binom{r}{4} + 92\binom{r}{5}\;. \tag{5.13}$$

In the very same way that we obtained (5.13) we get

$$\sum_{a,b,c,d=0}^{r} \max(a+b,\, c+d) = 22\binom{r}{1} + 170\binom{r}{2} + 420\binom{r}{3} + 420\binom{r}{4} + 148\binom{r}{5}, \tag{5.14}$$

$$\sum_{a,b,c,d=0}^{r} \min(a+b+c,d) = 7\binom{r}{1} + 61\binom{r}{2} + 159\binom{r}{3} + 164\binom{r}{4} + 59\binom{r}{5} \tag{5.15}$$

and

$$\sum_{a,b,c,d=0}^{r} \max(a+b+c,d) = 25\binom{r}{1} + 199\binom{r}{2} + 501\binom{r}{3} + 508\binom{r}{4} + 181\binom{r}{5}\;. \tag{5.16}$$

6. The author wishes to express his appreciation to Prof. L. Carlitz for suggestions which resulted in simplifications of several areas of this work and to Worden J. Updyke, Jr. for the programming and computing necessary in establishing (5.13), (5.14), (5.15), (5.16). The computations were performed on an IBM 360 model 20 at the Electronic Computer Programming Institute, Greensboro, North Carolina. Computer time was provided by the Institute.

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