Now, (5) and (7) imply that $t = n$ and $L_t = m$, and (6) and (8) are never simultaneously true. Thus $t \geqslant n$, with equality only if $L_n = m$. By the lemma,

$$k(m) = 2t = 2n$$

if and only if n and t are odd and $L_n = m$. The conclusion of the theorem follows.

REFERENCES

1. John Vinson, "The Relation of the Period Modulo m to the Rank of Apparition of m in the Fibonacci Sequence," *The Fibonacci Quarterly,* Vol. 1, No. 2 (April, 1963), pp. 37–45.
2. D.D. Wall, "Fibonacci Series Modulo m," *Amer. Math. Monthly,* 67 (1960), pp. 525–532.

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[Continued from page 348.]

$$H_k(x) = \sum_{n=0}^{\infty} H_n^k x^n \qquad (H_k(0) = (H_o)^k = r^k)\ ,$$

where

$$H_o(x) = f_0(x) = \sum_{n=0}^{\infty} x^n = (1-x)^{-1}$$

and

$$H_1(x) = (r+sx)(1-x-x^2)^{-1}$$

are

$$(2)\qquad \begin{cases} (1-3x+x^2)H_2(x) = r^2 - s^2x - 2exH_0(-x) \\ (1-4x-x^2)H_3(x) = r^3 + s^3x - 3exH_1(-x) \\ (1-7x+x^2)H_4(x) = r^4 - s^4x + 2e^2xH_0(x) - 8exH_2(-x) \\ (1-11x-x^2)H_5(x) = r^5 + s^5x + 5e^2xH_1(x) - 15exH_3(-x)\ . \end{cases}$$

The general expression for the generating function is (see [3])

$$(3)\qquad (1-a_kx+(-1)^kx^2)H_k(x) = r^k - (-s)^kx + kx\sum_{j=1}^{[k/2]} \frac{(-1)^j}{j}\, e^j a_{kj} H_{k-2j}((-1)^j x)\ ,$$

where

$$(1-x-x^2)^{-j} = \sum_{k=2j}^{\infty} a_{kj}x^{k-2j}\ ,$$

that is, a_{kj} are generated by the j^{th} power of the generating function for Fibonacci numbers f_n. Note the occurrence in (3) of the Lucas numbers a_n.

FUNCTIONS ASSOCIATED WITH THE GENERATING FUNCTIONS

In the process of obtaining (3), we use

$$(4)\qquad g_k(x) = \sqrt{5}\,H_k(x) = \sum_{j=0}^{[k/2]} \binom{k}{j} e^j F_{k-2j}((-1)^j x) \qquad (F_0(x) = H_0(x))\ ,$$

where

$$F_k(x) = [(r-sb)a]^k(1-a^kx)^{-1} + [(sa-r)b]^k(1-b^kx)^{-1} \qquad (k = 1,2,3,\cdots)$$

and

$$a = \frac{1+\sqrt{5}}{2}, \qquad b = \frac{1-\sqrt{5}}{2} \qquad (a,b \text{ roots of } x^2-x-1 = 0),$$

leading to the general inverse

[Continued on page 354.]