A RAPID METHOD TO FORM FAREY FIBONACCI FRACTIONS

KRISHNASWAMI ALLADI
Vivekananda College, Madras 600004, India

One question that might be asked after discussing the properties of Farey Fibonacci fractions [1] is the following: Is there any rough and ready method of forming the Farey sequence of Fibonacci numbers of order F_n, given n, however large? The answer is in the affirmative, and in this note we discuss the method. To form a standard Farey sequence of arbitrary order is no easy job, for the exact distribution of numbers coprime to an arbitrary integer cannot be given. The advantage of the Farey sequence of Fibonacci numbers is that one has a regular method of forming $f\cdot f_n$ without knowledge of $f\cdot f_m$ for $m < n$. We demonstrate our method with $F_9 = 34$; that is, we form $f\cdot f_9$.

STEP 1: Write down in ascending order the points of symmetry—fractions with numerator 1. (We use Theorem 1.1 here.)

$$\frac{1}{34},\ \frac{1}{21},\ \frac{1}{13},\ \frac{1}{8},\ \frac{1}{5},\ \frac{1}{3},\ \frac{1}{2},\ \frac{1}{1}\ .$$

STEP 2: Take and interval $(1/2, 1/1)$. Write down successively as demonstrated the alternate members of the Fibonacci sequence in increasing magnitude beginning with 2, less than or equal to F_n, for a prescribed $f\cdot f_n$. This will give a sequence of denominators

$$\frac{1}{2},\ \frac{}{5},\ \frac{}{13},\ \frac{}{34}\ .$$

STEP 3: Choose the maximum number of the Fibonacci sequence $\leqslant F_n$ not written in Step 2, and with this number as starting point write down successively the alternate numbers of the Fibonacci sequence in descending order of magnitude until 1.

$$\frac{}{21},\ \frac{}{8},\ \frac{}{3},\ \frac{1}{1}\ .$$

STEP 4: Put these two sequences together, the latter written later. (Theorem 1.2 has been used.)

$$\frac{1}{2},\ \frac{}{5},\ \frac{}{13},\ \frac{}{34},\ \frac{}{21},\ \frac{}{8},\ \frac{}{3},\ \frac{1}{1}\ .$$

STEP 5: Use the fact that $f_{(r+k)n}, f_{(r-k)n}$ have same denominators (Theorem 1.1) to get the sequence of denominators in all other intervals.

$$\frac{}{21},\frac{1}{34},\frac{1}{21},\frac{}{34},\frac{1}{13},\frac{}{34},\frac{}{21},\frac{1}{8},\frac{}{21},\frac{}{34},\frac{}{13},\frac{1}{5},\frac{}{13},\frac{}{34},\frac{}{21},\frac{}{8},\frac{1}{3},\frac{}{8},\frac{}{21},\frac{}{34},\frac{}{13},\frac{}{5},\frac{1}{2},\frac{}{5},\frac{}{13},\frac{}{34},\frac{}{21},\frac{}{8},\frac{}{3},\frac{1}{1}$$

STEP 6: Use the concept of factor of an interval to form numerators. The numerators of $(1/2, 1/1)$ will differ in suffix one from the corresponding denominators. The numerators of $(1/3, 1/1)$ will differ by suffix 2 from the corresponding denominators, $\cdots$. Use the above to form numerators and hence the Farey sequence in $[0,1]$. The first fraction is $0/F_{n-1}$.

$$\frac{0}{21},\frac{1}{34},\frac{1}{21},\frac{2}{34},\frac{1}{13},\frac{3}{34},\frac{2}{21},\frac{1}{8},\frac{3}{21},\frac{5}{34},\frac{2}{13},\frac{1}{5},\frac{3}{13},\frac{8}{34},\frac{5}{21},$$

$$\frac{2}{8},\frac{1}{3},\frac{3}{8},\frac{8}{21},\frac{13}{34},\frac{5}{13},\frac{2}{5},\frac{1}{2},\frac{3}{5},\frac{8}{13},\frac{21}{34},\frac{13}{21},\frac{5}{8},\frac{2}{3},\frac{1}{1}\ .$$

To form the fractions in the intervals $(1,2), (2,3), (3,5), \cdots$, write the reciprocals in reverse order of the fractions in $(1/2, 1)$ in $f \cdot f_{n+1}$, of $(1/3, 1/2)$ in $f \cdot f_{n+2}, \cdots$, respectively. This gives $f \cdot f_n$ as far as we want it.

In fact, one of the purposes of investigating the symmetries of Farey Fibonacci sequences was to develop easy methods to form them.

REFERENCE

1. Krishnaswami Alladi, "A Farey Sequence of Fibonacci Numbers," *The Fibonacci Quarterly*, Vol. 13, No. 1 (Feb. 1975), pp.

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A SIMPLE PROOF THAT PHI IS IRRATIONAL

JEFFREY SHALLIT
Student, Lower Merion High School, Ardmore, Pennsylvania 19003

Most proofs of the irrationality of phi, the golden ratio, involve the concepts of number fields and the irrationality of $\sqrt{5}$. This proof involves only very simple algebraic concepts.

Denoting the golden ratio as ϕ, we have

$$\phi^2 - \phi - 1 = 0 .$$

Assume $\phi = p/q$, where p and q are integers with no common factors except 1. For if p and q had a common factor, we could divide it out to get a new set of numbers, p' and q'.

Then

$$(p/q)^2 - p/q - 1 = 0$$

$$(p/q)^2 - p/q = 1$$

$$p^2 - pq = q^2$$

$$p(p-q) = q^2 \tag{1}$$

Equation (1) implies that p divides q^2, and therefore, p and q have a common factor. But we already know that p and q have no common factor other than 1, and p cannot equal 1 because this would imply $q = 1/\phi$, which is not an integer. Therefore, our original assumption that $\phi = p/q$ is false and ϕ is irrational.

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