EULERIAN NUMBERS AND OPERATORS

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1. INTRODUCTION

The Eulerian numbers $A_{n,k}$ are usually defined by means of the generating function

$$\frac{1-y}{e^{x(y-1)}-y} = 1+\sum_{n=1}^{\infty} \frac{x^n}{n!} \sum_{k=1}^{n} A_{n,k}y^{k-1} \tag{1.1}$$

or equivalently

$$\frac{1-y}{1-ye^{x(1-y)}} = 1+\sum_{n=1}^{\infty} \frac{x^n}{n!} \sum_{k=1}^{n} A_{n,k}y^{k} . \tag{1.2}$$

From either generating function we can obtain the recurrence

$$A_{n+1,k} = (n-k+2)A_{n,k-1} + kA_{n,k} \tag{1.3}$$

and the symmetry relation

$$A_{n,k} = A_{n,n-k+1} . \tag{1.4}$$

For references see [5, pp. 487–491], [6], [7], [8, Ch. 8].

In an earlier expository paper [1] one of the writers has discussed algebraic and arithmetic properties of the Eulerian numbers but did not include any combinatorial properties. The simplest combinatorial interpretation is that A_{nk} is the number of permutations of

$$Z_n = \left\{ 1,2,\cdots,n \right\}$$

with k rises, where we agree to count a conventional rise to the left of the first element. Conversely if we define $A_{n,k}$ as the number of such permutations, the recurrence (1.3) and the symmetry relation (1.4) follow almost at once but it is not so easy to obtain the generating function.

The symmetry relation (1.4) is by no means obvious from either (1.1) or (1.2). This suggests the introduction of the following symmetrical notation:

$$A(r,s) = A_{r+s+1,s+1} = A_{r+s+1,r+1} = A(s,r). \tag{1.5}$$

It is then not difficult to verify that (1.1) implies

$$\sum_{r,s=0}^{\infty} A(r,s) \frac{x^r y^s}{(r+s+1)!} = \frac{e^x-e^y}{xe^y-ye^x} \tag{1.6}$$

from which the symmetry is obvious. Moreover there is a second generating function

$$\sum_{r,s=0}^{\infty} A(r,s) \frac{x^r y^s}{(r+s)!} = (1+xF(x,y))(1+yF(x,y)), \tag{1.7}$$

where

$$F(x,y) = \frac{e^x-e^y}{xe^y-ye^x} .$$

The generating function (1.7) suggests the following generalization.

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(1.8) $$\sum_{r,s=0}^{\infty} A(r,s|\alpha,\beta)\frac{x^ry^s}{(r+s)!} = (1+xF(x,y))^{\alpha}(1+yF(x,y))^{\beta},$$

where the parameters α,β are unrestricted. Clearly

$$A(r,s|1,1) = A(r,s)$$

and

$$A(r,s|\alpha,\beta) = A(s,r|\beta,\alpha).$$

Moreover $A(r,s|\alpha,\beta)$ satisfies the recurrence

(1.9) $$A(r,s|\alpha,\beta) = (r+\beta)A(r,s-1|\alpha,\beta)+(s+\alpha)A(r-1,s|\alpha,\beta).$$

It follows from (1.9) and $A(0,0|\alpha,\beta)=1$ that $A(r,s|\alpha,\beta)$ is a polynomial in α,β and that the numerical coefficients in this polynomial are positive integers. Algebraic properties of $A(r,s|\alpha,\beta)$ corresponding to the known properties of $A(r,s)$ have been obtained in [3]; also this paper includes a number of combinatorial applications. We shall give a brief account of these results in the present paper. Of the combinatorial applications we mention in particular the following two.

Let $P(r,s,k)$ denote the number of permutations of Z_{r+s-1} with r rises, s falls and k maxima; we count a conventional fall on the extreme right as well as a conventional rise on the left. We show

(1.10) $$P(r+1,s+1,k+1) = \binom{r+s-2k}{r-k} C(r+s,k),$$

where

(1.11) $$A(r,s) = \sum_{j=0}^{\min(r,s)} \binom{r+s-2j}{r-j} C(r+s,j);$$

$C(r+s,s)$ is equal to the number of permutations of Z_{r+s+1} with $r+1$ rises, $s+1$ falls and $s+1$ maxima. Also we obtain a generating function for $P(r,s,k)$.

The element a_k in the permutation $(a_1a_2\cdots a_n)$ is called a *left upper record* if

$$a_i < a_k \qquad (1 \leqslant i < k);$$

it is a *right upper record* if

$$a_i > a_k \qquad (k < i \leqslant n).$$

Let $A(r,s;t,u)$ denote the number of permutations with $r+1$ rises, $s+1$ falls, t left and u right upper records. Then we show that

(1.12) $$A(r,s|\alpha,\beta) = \sum_{t,u} A(r,s;t,u)\alpha^{t-1}\beta^{u-1},$$

so that the coefficients in the polynomial $A(r,s|\alpha,\beta)$ have a simple combinatorial description.

If we put

$$A_n(x,y|\alpha,\beta) = \sum_{r+s=n} A(r,s|\alpha,\beta)x^ry^s,$$

it follows from the recurrence (1.9) that

$$A_n(x,y|\alpha,\beta) = [\alpha x+\beta y+xy(D_x+D_y)]A_{n-1}(x,y|\alpha,\beta).$$

Hence

(1.13) $$A_n(x,y|\alpha,\beta) = [\alpha x+\beta y+xy(D_x+D_y)]^n\cdot 1.$$

Thus it is of interest to expand the operator

$$\Omega^n_{\alpha,\beta}[\alpha x+\beta y+xy(D_x+D_y)]^n.$$

We show that

(1.14) $$\Omega^n_{\alpha,\beta} = \sum_{k=0}^{n} C^{(\alpha,\beta)}_{n,k}(x,y)(xy)^k(D_x+D_y)^k,$$

where

(1.15) $$C^{(\alpha,\beta)}_{n,k}(x,y) = \frac{1}{k!(\alpha+\beta)_k}(D_x+D_y)^kA_n(x,y),$$

where

$$(\alpha+\beta)_k = (\alpha+\beta)(\alpha+\beta+1)\cdots(\alpha+\beta+k-1).$$

The case $\alpha+\beta$ equal to zero or a negative integer requires special treatment.

As an application of (1.9) we cite

$$A_{m+n}(x,y\,|\,\alpha,\beta) = \sum_{k=0}^{\min(m,n)} \frac{1}{k!(\alpha+\beta)_k}(xy)^k (D_x+D_y)^k A_m(x,y\,|\,\alpha,\beta)(D_x+D_y)^k A_n(x,y\,|\,\alpha,\beta). \tag{1.16}$$

For additional results see §8 below.

2. THE NUMBERS $A(r,s)$

Let

$$\pi = (a_1 a_2 \cdots a_n)$$

denote an arbitrary permutation of Z_n. A *rise* is a pair of consecutive elements a_i, a_{i+1} such that $a_i < a_{i+1}$; a *fall* is a pair a_i, a_{i+1} such that $a_1 > a_{i+1}$. In addition we count a conventional rise to the left of a_1 and a conventional fall to the right of a_n. If π has $r+1$ rises and $s+1$ falls, it is clear that

$$r+s = n+1. \tag{2.1}$$

Let $A(r,s)$ denote the number of permutations of Z_{r+s+1} with $r+1$ rises and $s+1$ falls. Let π be a typical permutation with $r+1$ rises and $s+1$ falls and consider the effect of inserting the additional element $n+1$. If it is inserted in a rise, the number of rises remains unchanged while the number of falls is increased by one; if it is inserted in a fall, the number of rises is increased by one while the number of falls is unchanged. This implies

$$A(r,s) = (r+1)A(r,s-1)+(s+1)A(r-1,s). \tag{2.2}$$

Next if $\pi = (a_1a_2\cdots a_n)$ and we put

$$b_i = n-a_i+1 \qquad (i = 1,2,\cdots,n),$$

then corresponding to the permutation π we get the permutation

$$\pi' = (b_1b_2\cdots b_n)$$

which has $r+1$ falls and $s+1$ rises. It follows at once that

$$A(r,s) = A(s,r). \tag{2.3}$$

Another recurrence that is convenient for obtaining a generating function is

$$A(r,s) = A(r,s-1)+A(r-1,s)+\sum_{j<r}\sum_{k<s}\binom{r+s}{j+k+1}A(j,k)A(r-j-1,s-k-1). \tag{2.4}$$

This recurrence is obtained by deleting the element $r+s+1$ from a typical permutation with $r+1$ rises and $s+1$ falls.

Now put

$$F(z) = \sum_{r,s=0}^{\infty} A(r,s)\frac{x^ry^sz^{r+s+1}}{(r+s+1)!}. \tag{2.5}$$

By (2.4)

$$\sum_{r,s=0}^{\infty} A(r,s)\frac{x^ry^sz^{r+s}}{(r+s)!} = 1+\sum_{r,s=0}^{\infty} A(r,s)\frac{x^ry^{s+1}z^{r+s+1}}{(r+s+1)!}+\sum_{r,s=0}^{\infty} A(r,s)\frac{x^{r+1}y^sz^{r+s+1}}{(r+s+1)!}$$
$$+\sum_{j,k=0}^{\infty} A(j,k)\frac{x^jy^kz^{j+k+1}}{(j+k+1)!}\sum_{r,s=0}^{\infty} A(r,s)\frac{x^{r+1}y^{s+1}z^{r+s+1}}{(r+s+1)!}.$$

This implies

$$F'(z) = 1+(x+y)F+xyF^2. \tag{2.6}$$

Since $F(0) = 1$, it is easily verified that the differential equation (2.6) has the solution

$$F(z) = \frac{e^{xz}-e^{yz}}{xe^{yz}-ye^{xz}}.$$

Hence, taking $z = 1$, we get the generating function

(2.7) $$\frac{e^x - e^y}{xe^y - ye^x} = \sum_{r,s=0}^{\infty} A(r,s) \frac{x^r y^s}{(r+s+1)!} .$$

It is convenient to put

(2.8) $$F = F(x,y) = \frac{e^x - e^y}{xe^y - ye^x} .$$

It is easily verified that

(2.9) $$(D_x + D_y)F = F^2 ,$$

(2.10) $$(1 + xD_x + yD_y)F = (1 + xF)(1 + yF),$$

where $D_x = \partial/\partial x,\ D_y = \partial/\partial y$.

It is evident from (2.7) that

$$(1 + xD_x + yD_y)F = \sum_{r,s=0}^{\infty} A(r,s) \frac{x^r y^s}{(r+s)!} .$$

We therefore have the second generating function

(2.11) $$(1 + xF(x,y))(1 + yF(x,y)) = \sum_{r,s=0}^{\infty} A(r,s) \frac{x^r y^s}{(r+s)!} .$$

We note that iteration of (2.9) gives

(2.12) $$(D_x + D_y)^k F = k!F^{k+1} .$$

3. GENERALIZED EULERIAN NUMBERS

Put

(3.1) $$\Phi_{\alpha;\beta} = \Phi_{\alpha,\beta}(x,y) = (1 + xF(x,y))^{\alpha}(1 + yF(x,y))^{\beta}$$

and define $A(r,s|\alpha,\beta)$ by means of

(3.2) $$\Phi_{\alpha,\beta} = \sum_{r,s=0}^{\infty} A(r,s|\alpha,\beta) \frac{x^r y^s}{(r+s)!} .$$

Then we have

$$A(r,s|1,1) = A(r,s) ,$$

$$A(r,s|1,0) = A(r-1,s), \qquad A(r,s|0,1) = A(r,s-1) ,$$

(3.3) $$A(r,s|\alpha,\beta) = A(s,r|\beta,\alpha),$$

also

(3.4) $$A(r,o|\alpha,\beta) = \alpha^r, \qquad A(o,s|\alpha,\beta) = \beta^s .$$

It is easily verified that

(3.5) $$(D_x + D_y)\Phi_{\alpha,\beta} = (\alpha+\beta)F\Phi_{\alpha,\beta}$$

and generally

(3.6) $$(D_k + D_y)^k \Phi_{\alpha,\beta} = (\alpha+\beta)_k F^k \Phi_{\alpha,\beta} ,$$

where

$$(\alpha+\beta)_k = (\alpha+\beta)(\alpha+\beta+1)\cdots(\alpha+\beta+k-1).$$

In the next place we have

$$\begin{aligned}(xD_x + yD_y)\Phi_{\alpha,\beta} &= \alpha(1+xF)^{\alpha-1}(1+yF)^{\beta}(x + x^2D_x + xyD_y)F + \beta(1+xF)^{\alpha}(1+yF)^{\beta-1}(y + xyD_x + y^2D_y)F \\ &= [\alpha x + \beta y + (\alpha+\beta)xtF]\Phi_{\alpha,\beta} .\end{aligned}$$

Hence by (3.5)

(3.7) $$(xD_x + yD_y)\Phi_{\alpha,\beta} = [\alpha x + \beta y + xy(D_x + D_y)]\Phi_{\alpha,\beta} .$$

This yields the recurrence

(3.8) $$A(r,s|\alpha,\beta) = (r+\beta)A(r,s-1|\alpha,\beta) + (s+\alpha)A(r-1,s|\alpha,\beta) .$$

We can also show, after some manipulation, that

$$A(r,s|a+k,\beta) = \frac{k!}{(a+\beta)_k}\sum_{t=0}^{r}\binom{s+t}{t}\frac{(a+\beta+r)_{k-t}}{(k-t)!}A(r-t,s+t|a,\beta)\,. \tag{3.9}$$

If we take $s=0$ and make use of (3.4) we get

$$(a+k)^r\binom{a+\beta+k-1}{k} = \sum_{t=0}^{r}\binom{a+\beta+k+t-1}{k-r+t}A(t,r-t|a,\beta)\,. \tag{3.10}$$

If $a+\beta$ is a positive integer, Eq. (3.10) becomes

$$(a+x)^r\binom{a+\beta+x-1}{a+\beta-1} = \sum_{t=0}^{r}\binom{a+\beta+x+t-1}{a+\beta+r-1}A(t,r-t|a,\beta)\,. \tag{3.11}$$

For $a=\beta=1$, Eq. (3.11) reduces to the known formula

$$(x+1)^{r+1} = \sum_{t=0}^{r}\binom{x+t+1}{r+1}A(t,r-t) = \sum_{t=0}^{r}\binom{x+t+1}{r+1}A_{r+1,t+1}\,. \tag{3.12}$$

In order to get an explicit expression for $A(r,s|a,\beta)$ we take

$$1+xF = \frac{(x-y)e^x}{xe^y-ye^x}, \qquad 1+yF = \frac{(x-y)e^y}{xe^y-ye^x}\,.$$

Then

$$\Phi_{\alpha,\beta} = \frac{(x-y)^{\alpha+\beta}e^{\alpha x+\beta y}}{(xe^y-ye^x)^{\alpha+\beta}} = \left(\frac{x-y}{x-y-x(1-e^{y-x})}\right)^{\alpha+\beta}e^{\beta(y-x)} = \sum_{k=0}^{\infty}\frac{(a+\beta)_k}{k!}\frac{x^k}{(x-y)^k}(1-e^{y-x})^k e^{\beta(y-x)}$$

$$= \sum_{k=0}^{\infty}\frac{(a+\beta)_k}{k!}\frac{x^k}{(x-y)^k}\sum_{j=0}^{k}(-1)^j\binom{k}{j}e^{(\beta+j)(y-x)} = \sum_{n=0}^{\infty}\frac{(y-x)^n}{n!}\sum_{k=0}^{n}\frac{(a+\beta)_n}{k!}\frac{x^k}{(x-y)^k}\sum_{j=0}^{k}(-1)^j\binom{k}{j}(\beta+j)^n$$

$$= \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{k=0}^{n}\frac{(a+\beta)_k}{k!}\sum_{t=0}^{n-k}(-1)^t\binom{n-k}{t}y^{n-k-t}x^{k+t}\sum_{j=0}^{k}(-1)^j\binom{k}{j}(\beta+j)^n$$

$$= \sum_{r,s=0}^{\infty}\frac{x^ry^s}{(r+s)!}\sum_{j=0}^{r}(-1)^{r-j}(\beta+j)^{n+j}\sum_{k=j}^{r+s}\frac{(a+\beta)_k}{j!(k-j)!}\binom{r+s-k}{s}\,.$$

The sum on the extreme right is equal to

$$\binom{a+\beta+j-1}{j}\binom{a+\beta+r+s}{r-j},$$

so that

$$\Phi_{\alpha,\beta} = \sum_{r,s=0}^{\infty}\frac{x^ry^s}{(r+s)!}\sum_{j=0}^{r}(-1)^{r-j}\binom{a+\beta+j-1}{j}\binom{a+\beta+r+s}{r-j}(\beta+j)^{r+s}\,.$$

Therefore

$$A(r,s|a,\beta) = \sum_{j=0}^{r}(-1)^{r-j}\binom{a+\beta+j-1}{j}\binom{a+\beta+r+s}{r-j}(\beta+j)^{r+s}\,. \tag{3.13}$$

In view of (3.3) we have also

$$A(r,s|a,\beta) = \sum_{j=0}^{s}(-1)^{s-j}\binom{a+\beta+j-1}{j}\binom{a+\beta+r+s}{s-j}(a+j)^{r+s} \tag{3.14}$$

For $a=\beta=1$, Eq. (3.14) reduces to

$$A(r,s) = \sum_{j=0}^{s}(-1)^{s-j}\binom{r+s+2}{s-j}(j+1)^{r+s+1} = \sum_{j=1}^{s+1}(-1)^{s-j+1}\binom{r+s+2}{s-j+1}j^{r+s+1} \tag{3.15}$$

in agreement with a known formula for $A_{n,k}$.

Returning to the recurrence (3.8), iteration gives

$$A(r,s|\alpha,\beta) = (r+\beta)^2 A(r,s-2|\alpha,\beta) + [(r+\beta)(s+\alpha-1)+(s+\alpha)(r+\beta-1)]A(r-1,s-1|\alpha,\beta) + (s+\alpha)^2 A(r-2,s|\alpha,\beta).$$

This suggests a formula of the type

$$A(r,s|\alpha,\beta) = \sum_{j=0}^{k} B(j,k-j)A(r-j,s-k+j|\alpha,\beta) \qquad (0 \le k \le r+s), \tag{3.16}$$

where $B(j,k-j)$ depends also on r,s,α,β and is homogeneous of degree k in r,s,α,β. Applying (3.8) to (3.11) we get

$$B(j,k-j+1) = (r-j+\beta)B(j,k-j) + (s-k+j+\alpha-1)B(j-1,k-j+1).$$

Replacing k by $j+k-1$ this reduces to

$$B(j,k) = (r-j+\beta)B(j,k-1) + (s-k+\beta)B(j-1,k). \tag{3.17}$$

If we put

$$B(j,k) = (-1)^{j+k}\bar{B}(j,k),$$

(3.17) becomes

$$\bar{B}(j,k) = (j-r-\beta)\bar{B}(j,k-1) + (k-s-\alpha)\bar{B}(j-1,k). \tag{3.18}$$

Since, by (3.17),

$$B(j,0) = (r+\beta)^j, \qquad B(0,k) = (s+\alpha)^k,$$

it follows that

$$\bar{B}(j,0) = (-r-\beta)^j, \qquad \bar{B}(0,k) = (-s-\alpha)^k.$$

Hence

$$\bar{B}(j,k) = A(j,k|-s-\alpha,-r-\beta)$$

and (3.16) becomes

$$A(r,s|\alpha,\beta) = (-1)^k \sum_{j=0}^{K} A(j,k-j|-s-\alpha,-r-\beta)A(r-j,s-k+j|\alpha,\beta) \quad (0 \le k \le r+s). \tag{3.19}$$

For $k=r+s$ Eq. (3.19) reduces to

$$A(r,s|\alpha,\beta) = (-1)^{r+s}A(r,s|-s-\alpha,-r-\beta) \tag{3.20}$$

which can also be proved by using (3.13). Substituting from (3.20) in (3.19) we get

$$A(r,s|\alpha,\beta) = \sum_{j=0}^{k} A(j,k-j|s-k+j+\alpha,r-j+\beta)A(r-j,s-k+j|\alpha,\beta) \qquad (0 \le k \le r+s). \tag{3.21}$$

We remark that (3.21) is equivalent to

$$\Phi_{\alpha,\beta}\left\{x(1+z),y(1+z)\right\} = \Phi_{\alpha,\beta}\left\{x+xyzF(xz,yz),y+xyzF(xz,yz)\right\}\Phi_{\alpha,\beta}(xz,yz). \tag{3.22}$$

4. THE SYMMETRIC CASE

When $\alpha=\beta$ we define

$$A(r,s|\alpha) = A(r,s|\alpha,\alpha) = A(r,s|\alpha,\alpha) \tag{4.1}$$

and

$$\Phi_\alpha(x,y) = \Phi_{\alpha,\alpha}(x,y) = \Phi_\alpha(y,x).$$

Since $\Phi_\alpha(x,y)$ is symmetric in x,y we may put

$$\Phi_\alpha(x,y) = \sum_{n=0}^{\infty}\sum_{2j\le n} C(n,j|\alpha)\frac{(xy)^j(x+y)^{n-2j}}{n!}. \tag{4.2}$$

Since

$$(xD_x+yD_y)\Phi_\alpha = \alpha(x+y)\Phi_\alpha + xy(D_x+D_y)\Phi_\alpha$$

and

$$(xD_x+yD_y)\Phi_\alpha = \sum_{n=1}^{\infty}\sum_{2j\le n} C(n,j|\alpha)\frac{(xy)^j(x+y)^{n-2j}}{(n-1)!},$$

$$(x+y)\Phi_\alpha = \sum_{n=1}^{\infty} \sum_{2j<n} C(n-1,j|\alpha) \frac{(xy)^j (x+y)^{n-2j}}{(n-1)!},$$

$$xy(D_x+D_y)\Phi_\alpha = \sum_{n=1}^{\infty} \sum_{2j\leqslant n} C(n-1,j-1|\alpha) \frac{2(n-2j)(xy)^j (x+y)^{n-2j}}{(n-1)!} + \sum_{n=1}^{\infty} \sum_{2j<n} C(n-1,j|\alpha) \frac{j(xy)^j (x+y)^{n-2j}}{(n-1)!},$$

it follows that

(4.3) $$C(n,j|\alpha) = 2(n-2j+1)C(n-1,j-1|\alpha) + (\alpha+j)C(n-1,j|\alpha).$$

The special case

(4.4) $$F(x,y) = \sum_{n=0}^{\infty} \sum_{2n\leqslant j} C(n,j) \frac{(xy)^j (x+y)^{n-2j}}{n!}$$

is of interest. It is easily seen that

(4.5) $$C(n,j) = C(n,j|1).$$

In the next place it follows from (4.2) that

(4.6) $$A(r,s|\alpha) = \sum_{j=0}^{\min(r,s)} \binom{r+s-2j}{r-j} C(r+s,j|\alpha)$$

and in particular, for $\alpha = 1$,

(4.7) $$A(r,s) = \sum_{j=0}^{\min(r,s)} \binom{r+s-2j}{r-j} C(r+s,j).$$

To invert (4.7) we use the identity

$$x^n + y^n = \sum_{2j\leqslant n} (-1)^j \frac{n}{n-j} \binom{n-j}{j} (xy)^j (x+y)^{n-2j}.$$

We find that

(4.8) $$\begin{cases} C(n,k|\alpha) = \displaystyle\sum_{r=0}^{r} (-1)^{k-r} \frac{n-2r}{n-k-r} \binom{n-k-r}{k-r} A(r,n-r|\alpha) & (n \neq 2k), \\ C(2k,k|\alpha) = 2 \displaystyle\sum_{r=0}^{k-1} (-1)^{k-r} A(r,2k-r|\alpha) + A(k,k|\alpha). \end{cases}$$

To get a generating function for $C(n,j|\alpha)$ put $u = x+y$, $v = xy$ in (4.2). We get after some manipulation

(4.9) $$\sum_{n,j=0}^{\infty} C(n+2j,j|\alpha) \frac{u^n v^j}{(n+2j)!} = \left\{ \cosh \tfrac{1}{2}\sqrt{u^2-4v} - u\, \frac{\sinh \frac{1}{2}\sqrt{u^2-4v}}{\sqrt{u^2-4v}} \right\}^{-2\alpha}$$

The following values of $A(r,s)$, $C(n,j)$ are easily computed.

$A(r,s)$

1					
1	1				
1	4	1			
1	11	11	1		
1	26	66	26	1	
1	57	302	302	57	1

$C(n,j)$

1		
1		
1	2	
1	8	
1	22	16
1	52	136

5. ENUMERATION BY RISES, FALLS AND MAXIMA

We consider first the enumeration of permutations by number of maxima. Let $M(n,k)$ denote the number of permutations of Z_n with k maxima. Since we count a conventional fall on the right there is no ambiguity in counting the number of maxima. For example the permutation (1243) has one maxima while (3241) has two.

Let π denote an arbitrary permutation of Z_n with k maxima. If the element $n+1$ is inserted immediately to the left or right of a maximum the number of maxima does not change. If however it is inserted in any other position, the number of maxima becomes $k+1$. Therefore we have

(5.1) $$M(n+1,k) = (n-2k+3)M(n,k-1) + 2kM(n,k).$$

If we put

$$M(n,k) = 2^{n-2k+1}\overline{M}(n,k),$$

(5.1) becomes

(5.2) $$\overline{M}(n+1,k) = 2(n-2k+3)\overline{M}(n,k-1) + k\overline{M}(n,k) \qquad (1 \leqslant k \leqslant n) .$$

If we take $a = 1$ in (4.3) we get

(5.3) $$C(n,j) = 2(n-2j+1)C(n-1,j-1) + (j+1)C(n-1,j) \qquad (0 \leqslant j \leqslant n) .$$

It follows that

$$\overline{M}(n+1,k+1) = C(n,k),$$

so that

(5.4) $$M(n+1,k+1) = 2^{n-2k}C(n,k).$$

Thus (4.9) yields the generating function

(5.5) $$\sum_{n,j=0}^{\infty} M(n+2j+1,j+1)\,\frac{u^n v^j}{(n+2j)!} = \left\{ \cosh\sqrt{u^2-v} - \frac{u}{\sqrt{u^2-v}}\sinh\sqrt{u^2-v} \right\}^{-2} .$$

This result may be compared with [4].

We now consider the enumeration of permutations by rises, falls and maxima. Let $P(r,s,k)$ denote the number of permutations with r rises, s falls and k maxima, subject to the usual conventions. Let π be an arbitrary permutation with r rises, s falls and k maxima and consider the effect of inserting the additional element $r+s$. There are four possibilities depending on the location of the new element.

(i) immediately to the right of a maximum:

$$r \to r+1, \quad s \to s, \quad k \to k;$$

(ii) Immediately to the left of a maximum:

$$r \to r, \quad s \to s+1, \quad k \to k;$$

(iii) in any other rise:

$$r \to r, \quad s \to s+1, \quad k \to k+1;$$

(iv) in any other fall:

$$r \to r+1, \quad s \to s, \quad k \to k+1 .$$

We accordingly get the recurrence

(5.6) $$P(r,s,k) = kP(r-1,s,k) + kP(r,s-1,k) + (r-k+1)P(r,s-1,k-1) + (s-k+1)P(r-1,s,k-1) .$$

It is convenient to put

(5.7) $$P(r,s,k) = \binom{r+s-2k}{r-k} B(r,s,k).$$

Then (5.6) becomes

(5.8) $$B(r,s,k) = \frac{k(r-k)}{r+s-2k}\,B(r-1,s,k) + \frac{k(s-k)}{r+s-2k}\,B(r,s-1,k)$$
$$+ (r+s-2k+1)(B(r-1,s,k-1) + B(r,s-1,k)) .$$

We then show by induction that

$$B(r,s,k) = \phi(r+s,k),$$

that is, $B(r,s,k)$ is a function of $r+s$ and k. Indeed we show that

(5.9) $$B(r+1,s+1,k+1) = C(r+s,k),$$

where $C(r+s,k)$ has the same meaning as in (5.3).

Substituting from (5.9) in (5.7) we get

(5.10) $$P(r+1,s+1,k+1) = \binom{r+s-2k}{r-k} C(r+s,k).$$

It follows from (5.10) that

$$M(n+1,k+1) = \sum_{r+s=n} P(r+1,s+1,k+1) = \sum_{r+s=n} \binom{r+s-2k}{r-k} C(r+s,k) = 2^{n-2k}C(n,k)$$

in agreement with (5.4)

We remark that for $r = s = k$

(5.11) $$P(k+1,k+1,k+1) = C(2k,k) = A(2k+1),$$

the number of down-up (or up-down) permutations of Z_{2k+1}. It is well known that

$$\sum_0^\infty A(2k+1)\frac{x^{2k+1}}{(2k+1)!} = \tan x. \tag{5.12}$$

Generating functions for $P(r,s,k)$ are furnished by

$$\sum_{r,s=0}^\infty \sum_{k=0}^{\min(r,s)} P(r+1,s+1,k+1)\frac{x^r y^s z^k}{(r+s+1)!} = F(U,V), \tag{5.13}$$

and

$$\sum_{r,s=0}^\infty \sum_{k=0}^{\min(r,s)} P(r+1,s+1,k+1)\frac{x^r y^s z^k}{(r+s)!} = (1+UF(U,V))(1+VF(U,V)), \tag{5.14}$$

where

$$\left\{\begin{array}{l} U = \tfrac12(x+y+\sqrt{(x+y)^2-4xyz}) \\ V = \tfrac12(x+y-\sqrt{(x+y)^2-4xyz}) \end{array}\right. \tag{5.15}$$

and

$$F(U,V) = \frac{e^U - e^V}{Ue^V - Ve^U}.$$

6. (α,β)-SEQUENCES

Let α,β be fixed positive integers. We shall generalize rises, falls and maxima in the following way. In addition to the "real" elements $1, 2, \cdots, n$ we introduce two kinds of "virtual" elements which will be denoted by the symbols 0, 0′. There are α symbols 0 and β symbols 0′. To begin with $(n = 1)$ we have

$$\underbrace{0\cdots 0}_{\alpha}\ 1\ \underbrace{0'\cdots 0'}_{\alpha}. \tag{6.1}$$

We then insert the symbols $2, 3, \cdots, n$ in all possible ways subject to the requirement that there is at least one 0 on the extreme left and at least one 0′ on the extreme right. The resulting sequence is called an (α,β)-sequence. A rise is defined as a pair of consecutive elements a, b with $a < b$; here a may be 0. A fall is as a pair of consecutive elements a, b with $a > b$; now b may be 0′. The element b is a maximum if a, b, c are consecutive and a,b is a rise while b, c is a fall. For example in

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we have

$$\alpha = 2,\quad \beta = 3,\quad r = 4,\quad s = 3,\quad k = 1.$$

Let $P(r,s,k\,|\,\alpha,\beta)$ denote the number of (α,β)-sequences with r rises, s falls and k maxima. Then we have the recurrence

$$\begin{aligned} P(r,s,k\,|\,\alpha,\beta) &= (k+-1)P(r-1,s,k\,|\,\alpha,\beta) + (k+-1)P(r,s-1,k\,|\,\alpha,\beta) \\ &\quad + (r-k+1)P(r,s-1,k-1\,|\,\alpha,\beta) + (s-k+1)P(r-1,s,k-1\,|\,\alpha,\beta). \end{aligned} \tag{6.2}$$

In the special case $\alpha = \beta$ we put

$$P(r,s,k\,|\,\alpha) = P(r,s,k\,|\,\alpha,\alpha). \tag{6.3}$$

We also put

$$P(r,s,k\,|\,\alpha) = \binom{r+s-2k}{r-k} Q(r,s,k\,|\,\alpha). \tag{6.4}$$

Now let $M(n,k\,|\,\alpha,\beta)$ denote the number (α,β)-sequences with n real elements and k maxima. Then we have the recurrence

$$\begin{aligned} M(n+1,k\,|\,\alpha,\beta) &= (2k+\alpha+\beta-2)M(n,k\,|\,\alpha,\beta) \\ &\quad + (n-2k+3)M(n,k-1\,|\,\alpha,\beta). \end{aligned} \tag{6.5}$$

In particular, for

$$M(n,k\,|\,\alpha) = M(n,k\,|\,\alpha,\alpha),$$

(6.5) reduces to

$$M(n+1,k\,|\,\alpha) = 2(k+\alpha-1)M(n,k\,|\,\alpha) + (n-2k+3)M(n,k-1\,|\,\alpha). \tag{6.6}$$

We find that

$$M(n+1,k+1\,|\,\alpha) = 2^{n-2k}C(n,k\,|\,\alpha), \tag{6.7}$$

and

$$Q(r+1,s+1,k+1\,|\,\alpha) = C(r+s,k\,|\,\alpha). \tag{6.8}$$

Hence, by (6.4) and (6.8),

(6.9) $$P(r+1, s+1, k+1|\alpha) = \binom{r+s-2k}{r-2k} C(r+s, k|\alpha).$$

A generating function for $P(r+1, s+1, k+1|\alpha)$ is given bv

(6.10) $$\sum_{r,s=0}^{\infty} \sum_{k=0}^{\min(r,s)} P(r+1, s+1, k+1|\alpha) \frac{x^r y^s z^k}{(r+s)!} = (1+UF(U,V))^{\alpha}(1+VF(U,V))^{\beta},$$

where U, V are given by (5.15).

For a generating function for $P(r+1, s+1, k+1|\alpha,\beta)$ see [3].

7. UPPER RECORDS

Returning to ordinary permutations, let $\pi = (a_1 a_2 \cdots a_n)$ be a permutation of Z_n. The element a_k is called a *left upper record* if

$$a_i < a_k \qquad (1 \leqslant i < k);$$

it is called a *right upper record* if

$$a_k > a_i \qquad (k < i \leqslant n).$$

Let $A(r, s; t, u)$ denote the number of permutations with $r+1$ rises, $s+1$ falls, t left and u right upper records. We make the usual conventions about rises and falls. Also let $A(r, s; t)$ denote the number of permutations with $r+1$ rises, $s+1$ falls and t left upper records; let $\bar{A}(r, s, u)$ denote the number of permutations with $r+1$ rises, $s+1$ falls and u right upper records.

To begin with we have

(7.1) $$A(r,s;t+1) = \sum_{j=0}^{r-1}\sum_{k=0}^{s-1} \binom{r+s}{j+k+1} A(j,k;t)A(r-j-1, s-k-1) + A(r-1,s;t) \qquad (t > 0)$$

and

(7.2) $$A(r,s;1) = A(r,s-1) \qquad (s \geqslant 1).$$

Put

$$F_t(z) = \sum_{r,s=0}^{\infty} A(r,s;t) \frac{x^r y^s z^{r+s+1}}{(r+s+1)!}.$$

Then, for $t > 0$,

$$F'_{t+1}(z) = \sum_{r,s=0}^{\infty} A(r,s;t) \frac{x^{r+1} y^s z^{r+s+1}}{(r+s+1)!} + \sum_{j,k=0}^{\infty} A(j,k;t) \frac{x^j y^k z^{j+k+1}}{(j+k+1)!} \cdot \sum_{r,s=0}^{\infty} A(r,s) \frac{x^{r+1} y^{s+1} z^{r+s+1}}{(r+s+1)!},$$

so that

(7.3) $$F'_{t+1}(z) = F_t(z)(x + xyF(z)),$$

where

$$F(z) = \frac{e^{xz} - e^{yz}}{xe^{yz} - ye^{xz}}.$$

Also, by (7.2),

(7.4) $$F'_1(z) = 1 + yF(z).$$

If we put

$$G(z) = \sum_{t=1}^{\infty} F_t(z)\lambda^t,$$

it follows from (7.3) and (7.4) that

$$G'(z) = \lambda G(z)(x + xyF(z)) + \lambda(1 + yF(z)).$$

The solution of this differential equation is

(7.5) $$G(z) = \frac{1}{x}\left\{(1+xF(z))^{\lambda} - 1\right\}.$$

Similarly if we put

$$\bar{F}_u(z) = \sum_{r,s=0}^{\infty} \bar{A}(r,s;u) \frac{x^r y^s z^{r+s+1}}{(r+s+1)!}, \qquad \bar{G}(z) = \sum_{u=1}^{\infty} \bar{F}_u(z)\lambda^u,$$

we have

(7.6)
$$\bar{G}(z) = \frac{1}{y}\left\{(1+yF(z))^{\lambda} - 1\right\} .$$

We now consider the general case. It follows from the definition that

(7.7)
$$A(r,s;t+1,u+1) = \sum_{j,k} \binom{r+s}{j+k+1} A(j,k;t)\bar{A}(r-j-1,s-k-1;u) \qquad (t>0,\ u>0)$$

and
$$\begin{cases} A(r,s;1,u+1) = \bar{A}(r,s-1;u) & (s>0,\ u>0) \\ A(r,s;t+1,1) = A(r-1,s;t) & (r>0,\ t>0) \end{cases} .$$

Now put
$$F_{t,u}(z) = \sum_{r,s=0}^{\infty} A(r,s;t,u)\frac{x^r y^s z^{r+s}}{(r+s)!} .$$

Then
$$\begin{cases} F'_{t+1,u+1}(z) = xyF_t(z)\bar{F}_u(z) & (t>0,\ u>0) \\ F'_{1,u+1}(z) = y\bar{F}_u(z) & (u>0) \\ F'_{t+1,1}(z) = xF_t(z) & (t>0) \\ F'_{1,1}(z) = 1 \end{cases}$$

Therefore, by (7.5) and (7.6),
$$\sum_{t,u=1}^{\infty} \alpha^t\beta^u \sum_{r,s=0}^{\infty} A(r,s;t,u)\frac{x^r y^s z^{r+s+1}}{(r+s+1)!} = \alpha\beta + \alpha\beta[(1+xF(z))^{\alpha} - 1] + \alpha\beta[(1+yF(z))^{\beta} - 1]$$
$$+ \alpha\beta[(1+xF(z))^{\alpha} - 1][(1+yF(z))^{\beta} - 1] = \alpha\beta(1+xF(z))^{\alpha}(1+yF(z))^{\beta} .$$

Taking $z = 1$ we get

(7.8)
$$\sum_{t,u=1}^{\infty} \alpha^t\beta^u \sum_{t,s=0}^{\infty} A(r,s;t,u)\frac{x^r y^s}{(r+s)!} = \alpha\beta(1+xF(x,y))^{\alpha}(1+yF(x,y))^{\beta} ,$$

where
$$F(x,y) = \frac{e^x - e^y}{xe^y - ye^x} .$$

It follows that

(7.9)
$$A(r,s|\alpha,\beta) = \sum_{t,u} A(r,s;t,u)\alpha^{t-1}\beta^{u-1} .$$

Thus the generalized Eulerian number $A(r,s|\alpha,\beta)$ has the explicit polynomial expansion (7.9).

If we put
$$R(n+1;t,u) = \sum_{r+s=n+1} A(r,s;t,u)$$
it is evident that $R(n+1;t,u)$ is the number of permutations of Z_{n+1} with t left and u right upper records. By taking $y = x$ in (7.8) we find that

(7.10)
$$R(n+1;t+1,u+1) = \binom{t+u}{t} S_1(n,t+u) ,$$

where $S_1(n,t+u)$ denotes a Stirling number of the first kind.

In particular, if we put
$$R(n+1;t) = \sum_{r+s=n} A(r,s;t), \qquad \bar{R}(n+1;t) = \sum_{r+s=n} \bar{A}(r,s;t) ,$$

we get

(7.11)
$$R(n;t) = \bar{R}(n;t) = S_1(n,t) .$$

It is easy to give a direct proof of (7.11).

8. EULERIAN OPERATORS

Put

(8.1)
$$A_n(x,y) = \sum_{r+s=n} A(r,s)x^r y^s .$$

It follows from recurrence (2.2) that

$$A_n(x,y) = (x+y+xy(D_x+D_y))A_{n-1}(x,y). \tag{8.2}$$

Iteration of (8.2) gives

$$A_n(x,y) = (x+y+xy(D_x+D_y))^n\cdot 1. \tag{8.3}$$

It is accordingly of interest to consider the expansion of the operator

$$\Omega^n \equiv [x+y+xy(D_x+D_y)]^n . \tag{8.4}$$

We find that

$$\Omega^n = \sum_{k=0}^{n} C_{n,k}(x,y)(xy)^k(D_x+D_y)^k , \tag{8.5}$$

where

$$C_{n,k}(x,y) = \frac{1}{k!(k+1)!}(D_x+D_y)^k A_n(x,y). \tag{8.6}$$

More generally if we put

$$A_n(x,y|\alpha,\beta) = \sum_{r+s=n} A(r,s|\alpha,\beta)x^r y^s , \tag{8.7}$$

it follows from (3.8) that

$$A_n(x,y|\alpha,\beta) = [\alpha x+\beta y+xy(D_x+D_y)]A_{n-1}(x,y|\alpha,\beta). \tag{8.8}$$

Thus

$$A_n(x,y|\alpha,\beta) = [\alpha x+\beta y+xy(D_x+D_y)]^n\cdot 1, \tag{8.9}$$

so that it is of interest to expand the operator

$$\Omega^n_{\alpha,\beta} \equiv [\alpha x+\beta y+xy(D_x+D_y)]^n . \tag{8.10}$$

We find that

$$\Omega^n_{\alpha,\beta} = \sum_{k=0}^{n} C^{(\alpha,\beta)}_{n,k}(x,y)(xy)^k(D_x+D_y)^k , \tag{8.11}$$

where

$$C^{(\alpha,\beta)}_{n,k}(x,y) = \frac{1}{k!(\alpha+\beta)_k}(D_x+D_y)^k A_n(x,y|\alpha,\beta) \tag{8.12}$$

provided $\alpha+\beta$ is not equal to zero or a negative integer. Note that

$$\Omega = \Omega_{1,1}, \qquad C_{n,k}(x,y) = C^{(1,1)}_{n,k}(x,y).$$

As an application of (8.8) and (8.11) we have

$$A_{m+n}(x,y|\alpha,\beta) = \sum_{k=0}^{\min(m,n)} \frac{1}{k!(\alpha+\beta)_k}(xy)^k(D_x+D_y)^k A_m(x,y|\alpha,\beta)\cdot(D_x+D_y)^k\cdot A_n(x,y|\alpha,\beta), \tag{8.13}$$

where again $\alpha+\beta$ is not equal to zero or a negative integer.

When $\alpha=\beta=0$, (8.11) becomes

$$(xy(D_x+D_y))^n = \sum_{k=1}^{\infty} C^{(0,0)}_{n,k}(x,y)(xy)^k(D_x+D_y)^k \qquad (n \geq 1). \tag{8.14}$$

We find that

$$C^{(0,0)}_{n,k}(x,y) = \frac{1}{k!(k-1)!}(D_x+D_y)^{k-1}A_{n-1}(x,y) \qquad (1 \leq k \leq n). \tag{8.15}$$

The formula

$$C^{(\alpha,\beta)}_{n,k}(x,y) = \frac{1}{k!(k-1)!}\sum_{j=k}^{n}\binom{n}{r}(D_x+D_y)^{k-1}A_{r-1}(x,y)\cdot A_{n-r}(x,y|\alpha,\beta) \quad (1 \leq k \leq n) \tag{8.16}$$

holds for arbitrary α,β. When $\alpha=\beta=0$, (8.16) reduces to (8.15).

In the next place we consider the inverse of (8.11), that is, (8.17) $$(xy)^n(D_x+D_y)^n = \sum_{k=0}^{n} B_{n,k}^{(\alpha,\beta)}(x,y)\Omega_{\alpha,\beta}^{k}\ .$$

We find that

(8.18) $$(D_x+D_y)B_{n,k}^{(\alpha,\beta)}(x,y) = n(\alpha+\beta+n-1)B_{n-1,k}^{\alpha,\beta}(x,y)$$

and

(8.19) $$\sum_{n=0}^{\infty}\frac{u^n}{n!}\sum_{k=0}^{n} B_{n,k}^{(\alpha,\beta)}(x,y)(x-y)^k v^k = (1-xu)^{-\alpha-v}(1-yu)^{-\beta+v}\ .$$

In the special case $\alpha=\beta=0$ we put

(8.20) $$b_{n,k} = \frac{1}{(n-1)!}B_{n,k}^{(0,0)}(x,y) \qquad (n \geqslant 1).$$

Then we have

(8.21) $$b_{n,1} = \frac{x^n-y^n}{x-y} \equiv \sigma_n\ ,$$

(8.22) $$b_{n+1,2} = \sum_{j=1}^{n}\frac{1}{j}\sigma_j\sigma_{n-j+1}$$

and generally

(8.23) $$b_{n+1,k} = \sum_{j=k-1}^{n}\frac{1}{j}b_{j,k-1}\sigma_{n-j+1}\ .$$

This may also be written in the form

(8.24) $$b_{n+k,k} = \sum_{j=0}^{n}\frac{1}{j+k-1}b_{j+k-1,k-1}\sigma_{n-j+1}\ .$$

Thus for example

$$b_{n+3,n} = \sum_{0\leqslant i\leqslant j\leqslant n}\frac{1}{(i+1)(j+2)}\sigma_{i+1}\sigma_{j-i+1}\sigma_{n-j+1}$$

$$b_{n+4,n} = \sum_{0\leqslant i\leqslant j\leqslant k\leqslant n}\frac{1}{(i+1)(j+2)(k+3)}\sigma_{i+1}\sigma_{j-i+1}\sigma_{k-j+1}\sigma_{n-k+1}$$

and so on.

For proof of the formulas in this section the reader is referred to [2].

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