THE NUMBER OF ORDERINGS OF n CANDIDATES WHEN TIES ARE PERMITTED*

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In a competition it is customary to rank the candidates permitting ties and it is an interesting elementary combinatorial problem to find the number $\omega(n)$ of such orderings when there are n labelled candidates. $\omega(n)$ has curious properties.

Theorem 1. $\omega(n)$ is equal to $n!$ times the coefficient of x^n in the expansion of $(2-e^x)^{-1}$, that is,

$$\sum_{n=0}^{\infty} \frac{\omega(n)x^n}{n!} = \frac{1}{2-e^x} \tag{1}$$

if $\omega(0)$ is defined as 1.

By multiplying by $2-e^x$ and equating coefficients we obtain the recurrence relation

$$\omega(n) = \delta_0^n + \sum_{r=0}^{n-1} \binom{n}{r} \omega(r), \tag{2}$$

where $\delta_0^n = 1$ and $\delta_0^n = 0$ if $n \neq 0$ ("Kronecker's delta").

I mentioned (1) without proof in an appendix to Mayer and Good (1973). [It may be compared with Proposition XXIV in Whitworth (1901/1951) which states that the number of ways in which n different things can be distributed into not more than n indifferent parcels is $n!$ times the coefficient of x^n in the expansion of $exp\,(e^x)/e.$]

Proof. Let r denote the number of distinct *positions* in an ordering of n candidates; for example, if among five candidates two tied for the first place, one was "third," and the other two were "fourth and fifth equal" we would say that the number of distinct positions is 3. We shall prove that the number $g(n,r)$ of orderings of n candidates having just r distinct "positions" is equal to $n!$ times the coefficient of x^n in $(e^x-1)^r$. (This is Whitworth's Proposition XXII whose proof is different.) Equation (1) then follows from the identity

$$(2-e^x)^{-1} = \sum_{r=0}^{\infty} (e^x-1)^r .$$

When there are just r "positions" for the n candidates, let us adopt the unconventional terminology of calling these positions *first, second,* $\cdots$, r^{th} and let us imagine that, for a specific ordering, there are n_1 candidates who are first, n_2 who are second, $\cdots$, and n_r who are r^{th}, where necessarily

$$n_1 \geqslant 1,\ n_2 \geqslant 1, \cdots, n_r \geqslant 1,\ n_1+n_2+\cdots+n_r = n.$$

The sequence of numbers $n_1, n_2, \cdots, n_r$ can be regarded as defining the *structure* of an ordering that has just r "positions." The number of orderings having just this structure (which incidentally is clearly a multiple of $r!$) is equal to the number of ways of throwing n labelled objects into r pigeon holes in such a way that there are n_1 in the first pigeon hole, n_2 in the second one, and so on. But this is equal to the multinomial coefficient $n!/(n_1! \cdots n_r!)$ Hence $g(n,r)$ is equal to $n!$ times the coefficient of x^n in

*For some overlooked references, see Sloan (1973), p. 109.

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(3) $$\left(x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)\left(x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)\cdots\left(x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right),$$

where there are r factors. The reason for putting in the x's here is that they automatically take care of the constraint $n_1+\cdots+n_r=n$. Equation (1) then follows immediately.

Theorem 2.

(4) $$\omega(n) = \sum_{r=0}^{\infty} \frac{r^n}{2^{r+1}} .$$

Proof. We have

$$(2-e^x)^{-1} = 2^{-1}\sum_{r=0}^{\infty}\frac{e^{rx}}{2^r} \qquad (|x| < \log_e 2)$$

and the result follows at once from Theorem 1.

Theorem 3.

(5) $$\omega(n) = \sum_{r=0}^{n} r!S_n^{(r)} = \sum_{r=0}^{n} \Delta^r 0^n$$

(6) $$= \sum_{r=0}^{n}\left\{r^n - \binom{r}{1}(r-1)^n + \binom{r}{2}(r-2)^n - \cdots + (-1)^r 0^n\right\},$$

where $S_n^{(r)}$ is a Stirling integer (number) of the second kind defined, for example, by Abramowitz and Stegun (1964, p. 824) or David and Barton (1962, p. 294), and tabulated in these two books on pages 835 and 294, respectively, and more completely in Fisher and Yates (1953, p. 78). Another notation for $S_n^{(r)}$ is $S(n,r)$, e.g. Riordan (1958). We could define $S_n^{(r)}$ by

(7) $$r!S_n^{(r)} = \Delta^r 0^n .$$

(Note the conventions $0^0=1$, $S_n^{(0)}=0$ if $n \geqslant 1$, $S_0^{(0)}=1$.)

Proof. It follows either from the proof of Theorem 1, or from Whitworth's Proposition XXII, that the term corresponding to a given value of r is equal to the contribution to $\omega(n)$ arising from those orderings of the n candidates having just r "positions." Equations (5) and (6) then follow at once. The "incidental" remark in the proof of Theorem 1 shows that $S_n^{(r)}$ is an integer.

An alternative proof of Theorem 3 follows from Theorem 2 by using the relationship between ordinary powers and factorial powers,

(8) $$r^n = \sum_{m=0}^{n} S_n^{(m)} r(r-1)\cdots(r-m+1),$$

combined with the binomial theorem for negative integral powers.

Theorem 3 provides one way of computing $\omega(n)$, given tables of $S_n^{(r)}$. The calculations can be partly checked by the special case of (8),

(9) $$\sum_{r=0}^{n}(-1)^r r!S_n^{(r)} = \sum_{r=1}^{n}(-1)^r\Delta^r 0^n = (-1)^n .$$

Theorem 4.

(10) $$\omega(n) = \frac{n!}{2}\left\{\tfrac{1}{2}\delta_0^n + \sum_{m=-\infty}^{\infty}\frac{1}{(\log_e 2 + 2\pi i m)^{n+1}}\right\}$$

$$\omega(n) - \tfrac{1}{4}\delta_0^n = \frac{n!}{2}\left\{\frac{1}{(\log_e 2)^{n+1}} + 2\sum_{m=1}^{\infty} \frac{\cos[(n+1)\theta_m]}{[(\log_e 2)^2 + 4\pi^2 m^2]^{(n+1)/2}}\right\} \tag{11}$$

$$= n!(\log_2 e)^{n+1}\left\{\tfrac{1}{2} + \sum_{m=1}^{\infty} \cos^{n+1}\theta_m \cos[(n+1)\theta_m]\right\}, \tag{12}$$

where

$$\theta_m = \tan^{-1}(2\pi m \log_2 e)$$

and the sum in (8) is a Cauchy principal value when $n = 0$.

Corollary.

$$\omega(n) \sim n!\,(\log_2 e)^{n+1}/2 \tag{13}$$

when n tends to infinity.

This asymptotic formula gives the answer to the nearest integer (and hence exactly) when $n < 16$ (see Table 1). It is curious that $n!\,(\log_2 e)^{n+1}/2$ is within 1/ 50 of an odd integer, namely $\omega(n)$, when $2 \leqslant n \leqslant 13$. We can obtain $\omega(n)$ exactly by taking the series of Theorem 4 as far as the first term for which $m > n/(2\pi e)$.

Proof of Theorem 4. By, say Titchmarch (1932, p. 113),

$$(1 - e^{-z})^{-1} = \tfrac{1}{2} + \lim_{M\to\infty} \sum_{m=-M}^{m} \frac{1}{z + 2m\pi i},$$

where z is a real or complex number, not a multiple of $2\pi i$. Put $z = u - x$ and we can deduce that the coefficient of x^n in the power series expansion of $(1 - e^{x-u})^{-1}$ at $x = 0$ (when $Re(u) > o$) is

$$\tfrac{1}{2}\delta_0^n + \lim_{M\to\infty} \sum_{m=-M}^{m} \frac{1}{(u + 2m\pi i)^{n+1}}. \tag{14}$$

Theorem 4 follows on putting $u = \log_e 2$.

TABLE 1

Fractional part of $a_{n,0}$ (denoted by $\{a_{n,0}\}$), and the values of $a_{n,1}, a_{n,2}$, and $a_{n,3}$, where $a_{n,m}$ denotes the terms of formula (11). The sum column gives the total to be added to the integral part of $a_{n,0}$.

n	$\{a_{n,0}\}$	$a_{n,1}$	$a_{n,2}$	$a_{n,3}$	*Sum*
1	.0406844905	−0.0244239291	−0.0062750652	−0.0028030856	.007
2	.0027807072	−0.0025628988	−0.0001650968	−0.0000327956	.000020
5	.0015185164	−0.0014866887	−0.0000285616	−0.0000026000	.00000067
10	.0052710420	−0.0052693807	−0.0000016476	−0.0000000133	.0000000004
16	.5130767435	0.4869198735	0.0000033805	0.0000000025	1.0000000000
20	.5284857660	27.4714964238	0.0000178075	0.0000000028	28.0000000000
25	.4328539621	22480.5672001073	−0.0000540633	−0.0000000061	22481.0000000000

Theorem 5. (i) If $n \equiv n' \pmod{p-1}$, where $n \geqslant 1$, $n' \geqslant 1$, we have

$$\omega(n) \equiv \omega(n') \pmod{p}, \tag{15}$$

where p is any prime. (ii) If $n \equiv 0 \pmod{p-1}$, where $n \geqslant 1$, then

$$\omega(n) \equiv 0 \pmod{p}, \tag{16}$$

where p is any odd prime.

COMMENT. If we had defined $\omega(0) = 0$, Part (ii) would have been a special case of Part (i), but unfortunately the convention $\omega(n) = 1$ is more convenient for Theorems 2 and 3.

Proof. To prove Theorem 5 we first give the following properties of the differences of powers at zero.

Lemma.

(17) (i) $$\Delta^a 0^b = 0 \quad \text{if} \quad a > b \quad (a,b = 1,2,3,\cdots)$$

(18) (ii) $$\Delta^r 0^n \equiv \Delta^r 0^{n'} \pmod p \quad \text{if} \quad n \equiv n' \pmod{p-1},\ n \geqslant 1,\ n' \geqslant 1$$

(19) (iii) $$\Delta^r 0^n \equiv (-1)^{r-1} \pmod p \quad \text{if} \quad n \equiv 0 \pmod{p-1},\ r \neq 0,\ n \neq 0\,.$$

Equation (17) is a special case of the fact that the a^{th} difference of a polynomial of degree b is zero if $a < b$. To prove (18) we first note that

(20) $$\Delta^r 0^n = \begin{cases} r^n - \binom{r}{1}(r-1)^n + \cdots + r(-1)^{r-1}1^n & (r > 0, n > 0) \\ 0 \quad (r = 0,\ n > 0) \\ 1 \quad (r = n = 0)\,. \end{cases}$$

But, by Fermat's theorem,

$$a^n \equiv a^{n'} \pmod p\,,$$

so that (18) follows at once from (20). If $n \equiv 0 \pmod{p-1}, n \neq 0, r \neq 1$, it follows from (20) and Fermat's theorem that

$$\Delta^r 0^n \equiv 1 - \binom{r}{1} + \cdots + \binom{r}{r-1}(-1)^{r-1} \pmod p$$

and this gives (19) by the binomial theorem.

To deduce Theorem 5, we now see from Eq. (5) that

$$\omega(n) = \sum_{r=0}^{n} \Delta^r 0^n \equiv \sum_{r=0}^{n} \Delta^r 0^{n'} \pmod p$$

by (18). Hence, by (5), with n replaced by n',

$$\omega(n) \equiv \omega(n') + \sum_{r=n'+1}^{n} \Delta^r 0^{n'} = \omega(n')$$

by (17). To prove Part (ii), where $n \equiv 0 \pmod{p-1},\ n \neq 0$, we have

$$\omega(n) = \sum_{r=0}^{n} \Delta^r 0^n \equiv \sum_{r=1}^{n} (-1)^{r-1}$$

by (19), and this vanishes because n is even when p is odd.

SOME DEDUCTIONS FROM THEOREM 5

(a) Taking $p = 2$ in Part (i) we see that $\omega(n)$ is always odd.
(b) Given any odd prime p, there are an infinity of values for n for which p divides $\omega(n)$.
(c) When n is even, 3 divides $\omega(n)$.
(d) 59 divides $\omega(69)$ and 78803 divides $\omega(78813)$. (See the factorization of $\omega(11)$ in Table 2.)
(e) $2^{11213} - 1$ divides $\omega(2^{11213} - 2)$, but the division will never be done!
(f) $\omega(sp) \equiv \omega(s) \pmod p$ $(s = 1, 2, 3, \cdots)$. [Here, and in (f), $\cdots$, (k), p is any prime number.]
(g) $\omega(p) \equiv 1 \pmod p$. (Also deducible easily from (2).)
(h) $\omega(p^k) \equiv 1 \pmod p$ $(k = 1, 2, 3, \cdots)$.
(i) $\omega(2p^k) \equiv 3 \pmod p$ $(k = 1, 2, 3, \cdots)$.
(k) $\omega(3p^k) \equiv 13 \pmod p$ $(k = 1, 2, 3, \cdots)$.

In Table 2, some prime factorizations of $\omega(n)$ are shown, and (g) is also exemplified. Large primes seem to have a propensity to appear as factors of $\omega(n)$.

Conjecture 1. Part (i) of Theorem 5 shows that the sequence $\omega(1), \omega(2), \omega(3), \cdots$ has period $p - 1$ when p is a prime. It may be conjectured that it never has a shorter period (properly dividing $p - 1$). If this is true then the

TABLE 2

Some Values of $\omega(n)$, and Some Prime Factorizations of $\omega(n)$ and of $\omega(n) - 1$

The values of $\omega(n)$ were computed independently by (2), (6), and (11). x denotes any number known to have no factor less than 214500.

n	$\omega(n)$	Factorization of $\omega(n)$ into Primes	Factorization of $\omega(n) - 1$ into Primes
1	1	—	—
2	3	Prime	2
3	13	Prime	2·2·3
4	75	3·5·5	2·37
5	541	Prime	2·2·3·3·3·5
6	4683	3·7·223	2·2341
7	47293	Prime	2·2·3·7·563
8	5 45835	3·5·36389	2·272917
9	70 87261	Prime	2·2·3·5·41·43·67
10	1022 47563	3·11·41·75571	2·29·263·6703
11	16226 32573	59·349·78803	2·2·3·3·11·283·14479
12	2 80915 67595	3·5·7·13·20579903	2·2969·4730813
13	52 68583 48381	Prime	2·2·3·5·7·13·13·113·65687
14	1064 13429 70443	3·x	2·37·x
15	23028 31909 77853	13·23·23·271·123564559	2·2·3·59·325258744319
16	5 31565 46819 81355	3·5·17·x	2·3659·4327·167871689
17	130 37076 70291 35901	101·157·809·10162736677	2·2·3·3·5·5·17·37·131·3889·452041
18	3385 53466 32568 45323	3·7·19·x	2·251·x
19	92801 58731 93284 11133	59·13723·25579·4480956911	2·2·3·7·19·x
20	26 77687 79624 43842 03115	3·5·11·37·229·x	2·x
21	811 24824 99850 40738 81821	x	2·2·3·5·11·3083·x
22	25748 44419 80319 03845 44203	3·23·x	2·59·167·1489·x
23	8 54384 51336 74570 92945 80413	14081·x	2·2·3·3·3·23·71·271·6263·x
24	295 82791 21074 14547 26506 48875	3·5·5·5·7·13·x	2·31·43·x
25	10669 73654 38475 77582 55834 98141	x	2·2·3·5·7·7·13·x
26	4 00222 57598 44168 49248 61275 39083	3·31·149·2969·x	2·x
27	155 89776 39186 21623 24927 62262 53693	13·131·x	2·2·3·29·x
28	6297 56206 49500 66033 51837 39353 34635	3·5·29·37·x	2·71·18223·119173·x
29	2 63478 38526 30236 90020 89332 90445 76861	x	2·2·3·3·5·29·43·997·1447·38167·x
30	114 03568 79401 18804 83742 46419 61849 01963	3·7·11·31·4289·85247·x	2·x
31	5100 08036 57426 93884 30841 02407 59181 18973	41·523·12829·103291·x	2·2·3·7·11·31·73·127·269·150907·x
32	2 35451 54085 73489 66491 84490 63714 48554 76395	3·5·17·66643·x	2·109·151·x
33	112 09597 42203 05626 82674 94209 29300 68825 89981	137·x	2·2·3·5·17·x
34	5498 49040 77825 68486 24268 68390 30104 97501 04843	3·x	2·59·3209·x
35	2 77642 56952 89206 00263 03102 19593 68549 61635 84253	x	2·2·3·3·37·x
36	144 19928 09516 55469 62836 09781 09406 91758 35130 90155	3·5·7·7·13·19·37·449·36017·x	2·199·x

converse of Part (ii) would be true; that is p could divide $\omega(n)$ only if $n \equiv 0 \pmod{p-1}$. I have verified the conjecture for all primes less than 73, but I do *not* regard this as strong evidence. In fact I estimate that the probability that the conjecture would have survived the tests, if it is false, is about 0.18.

If this conjecture is true then we can deduce that $\omega(n)$ is never a multiple of n, for any integer n greater than 1. Since $\omega(n)$ is always odd we need consider only odd values of n. *Suppose* then that n divides $\omega(n)$ and let p be a prime factor of n. Let the highest power of p that divides n be p^m. By repeated application of (f) we have $\omega(n) \equiv \omega(n/p^m) \pmod p$, and therefore by the converse of Part (ii) of Theorem 5 (which is true if the conjecture is) we see that n/p^m is a multiple of $p-1$ and is therefore even. But n is odd by assumption and we have arrived at a contradiction. So the conjecture implies that n cannot divide $\omega(n)$.

Conjecture 2. Modulo 2, 4, 8, 16, 32, 64, 128, 256, 512, ⋯ the sequence $\{\omega(n)\}$ runs into cycles of lengths 1, 2, 2, 2, 2, 4, 8, 16, 32, ⋯. That is the period modulo 2^k appears to be 2^{k-4} when $k \geqslant 5$, and, for k = 1, 2, 3, 4 is 1, 2, 2, and 2. This conjecture would follow from the following one.

Conjecture 3. If $\omega(n)$ is expressed in the binary system as

$$a_{n0} + 2a_{n1} + 2^2 a_{n2} + 2^3 a_{n3} + \cdots,$$

then the sequence of r^{th} least significant digits, $a_{1r}, a_{2r}, a_{3r}, \cdots$ runs into a cycle whose lengths, for $r = 0, 1, 2, 3, 4, \cdots$ are respectively 1, 2, 2, 1, 2, 4, 8, 16, ⋯. That is, the period is 2^{r-3} for $r \geqslant 3$ and for $r = 0, 1, 2$ is $1, 2,$ and 2. This conjecture is formulated on the basis of the columns of Table 3.

Conjecture 4. If $\omega(n)$ is expressed in the scale of p, where p is an odd prime,

$$\omega(n) = b_{n0} + pb_{n1} + p^2 b_{n2} + \cdots$$

then the sequence $b_{1r}, b_{2r}, b_{3r}, \cdots$ runs into a cycle of length $p^r(p-1)$. This has been verified empirically for $p^{r+1} = 9, 27,$ and 25 (and $n \leqslant 36$). For $r = 0$ we know the result is true by Theorem 5, as we said before. A feasible conjecture is that the periods are never less than the ones stated.

Conjecture 5. Modulo p^r, where p is an odd prime, and $r \geqslant 1$, the sequence $\{\omega(n)\}$ runs into a cycle of length $p^{r-1}(p-1)$ and no less. This would follow from Conjecture 4. It generalizes Conjecture 1.

From Conjectures 2 and 5, if they are true, we can deduce that, modulo $m = 2^k p_1^{k_1} p_2^{k_2} \cdots$, the sequence $\{\omega(n)\}$ runs into a cycle of length

$$\begin{array}{ll} \phi(m) & \text{if } k = 0, 1, \text{ or } 2 \\ \phi(m)/2 & \text{if } k = 3 \\ \phi(m)/4 & \text{if } k = 4 \\ \phi(m)/8 & \text{if } k \geqslant 5, \end{array}$$

where ϕ denotes Euler's arithmetic function.

Conjecture 6. Parts of Conjectures 2 to 5 could perhaps be proved inductively, by using Eq. (2) combined with the use of m^{th} roots of unity.

Conjecture 7. For each n, $\omega(n)$ and $\omega(n+1)$ have no common factor, and the highest common factor of $\omega(n) - 1$ and $\omega(n+1) - 1$ is 2. This follows from Conjecture 1.

GENERALIZATION OF SOME OF THE RESULTS

The proof of Theorem 4 suggests correctly that several formulae that we have mentioned can be generalized by replacing $\log_e 2$ by u. By making this change we see that, in addition to (14), we have:

The coefficient of x^n in $(1 - e^{x-u})^{-1}$ (where $Re(u) > 0$) is equal to

$$\frac{1}{n!}\left(-\frac{d}{du}\right)^n \frac{1}{1-e^{-u}} \tag{21}$$

$$= \tfrac{1}{2}\delta_0^n + \frac{(-1)^n}{n!} \sum_{m=[n+1/2]}^{\infty} \frac{B_{2m} u^{2m-n-1}}{2m(2m-n-1)!} \quad (u < 2\pi) \tag{22}$$

TABLE 3

The Ten Least Significant Binary Digits a_{nr} of $\omega(n)$ $(n = 1, 2, \cdots, 36)$

$n \backslash r$	9	8	7	6	5	4	3	2	1	0
1										1
2									1	1
3							1	1	0	1
4				1	0	0	1	0	1	1
5	1	0	0	0	0	1	1	1	0	1
6	1	0	0	1	0	0	1	0	1	1
7	0	0	1	0	1	1	1	1	0	1
8	0	0	0	0	1	0	1	0	1	1
9	0	0	1	0	0	1	1	1	0	1
10	0	0	1	0	0	0	1	0	1	1
11	0	0	0	1	1	1	1	1	0	1
12	0	1	1	1	1	0	1	0	1	1
13	1	1	0	1	0	1	1	1	0	1
14	1	0	0	1	0	0	1	0	1	1
15	0	1	0	0	1	1	1	1	0	1
16	0	1	1	0	1	0	1	0	1	1
17	1	0	0	0	0	1	1	1	0	1
18	0	0	0	0	0	0	1	0	1	1
19	0	1	1	1	1	1	1	1	0	1
20	0	1	0	1	1	0	1	0	1	1
21	0	0	1	1	0	1	1	1	0	1
22	0	1	1	1	0	0	1	0	1	1
23	1	0	1	0	1	1	1	1	0	1
24	0	1	0	0	1	0	1	0	1	1
25	1	1	1	0	0	1	1	1	0	1
26	1	1	1	0	0	0	1	0	1	1
27	1	1	0	1	1	1	1	1	0	1
28	0	0	1	1	1	0	1	0	1	1
29	1	0	0	1	0	1	1	1	0	1
30	0	1	0	1	0	0	1	0	1	1
31	0	0	0	0	1	1	1	1	0	1
32	0	0	1	0	1	0	1	0	1	1
33	0	1	0	0	0	1	1	1	0	1
34	1	1	0	0	0	0	1	0	1	1
35	0	0	1	1	1	1	1	1	0	1
36	0	0	0	1	1	0	1	0	1	1
Period		32	16	8	4	2	1	2	2	1
Antiperiod		16	8	4	2	1	–	1	1	–

$$= \frac{1}{n!}\sum_{r=0}^{\infty} r^n e^{-ru} \qquad (Re(u) > 0) \tag{23}$$

$$= \frac{1}{n!} e^u \sum_{m=0}^{n} S_r^{(m)} m!(e^u - 1)^{-m-1} \tag{24}$$

$$= \frac{1}{n!} e^u \sum_{m=0}^{n} (e^u - 1)^{-m-1} \Delta^m 0^n \tag{25}$$

$$= \tfrac{1}{2}\delta_0^n + \sum_{m=-\infty}^{\infty} \frac{\cos\,[(n+1)\tan^{-1}(2\pi m/u)]}{(u^2 + 4\pi^2 m^2)^{(n+1)/2}} \;. \tag{26}$$

For example,

$$\frac{1}{7!}\sum_{r=0}^{\infty} r^7 e^{-r} = \frac{e}{7!}\sum_{m=0}^{7} (e-1)^{-m-1}\Delta^m 0^7 = 1.00000023$$

and the coefficients of $1, x, x^2, x^3, \cdots$ in $(1 - e^{x-1})^{-1}$ are respectively

$$1.58,\ 0.92,\ 0.9962,\ 1.0011,\ 1.00014,\ 0.999982,\ 0.9999957,\ 1.00000023,\ \cdots,$$

tending rapidly to 1.

Formula (26) is always very effective for summing the series

$$\sum_{r=0}^{\infty} r^n z^r$$

numerically when $|z|$ is close to 1.

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