# EMBEDDING A SEMIGROUP IN A RING

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Let $S$ be a set of arbitrary cardinality. For each element $s \in S$, define a function $a_s : S \to Z_2$ by

$$a_s(t) = \begin{cases} 0 & \text{if } s \neq t \\ 1 & \text{if } s = t \end{cases} .$$

Denote the set of all such functions by $X(S)$. There is obviously a 1-1 correspondence between $S$ and $X(S)$ by mapping $s \to a_s$.

Let $f : S \to S$ be an arbitrary map. Define a map $m_f : S \times S \to Z_2$ by

$$m_f(t,s) = \begin{cases} 1 & \text{if } f(s) = t \\ 0 & \text{otherwise} \end{cases} ,$$

and define a map $\bar{f} : X(S) \to X(S)$ by

$$\bar{f}(a_s)(v) = \sum_{u \in s} m_f(v,u) a_s(u) .$$

Clearly,

$$\bar{f}(a_s) = a_{f(s)} ,$$

and there is a 1-1 correspondence between $S^s$ = the set of all functions of $S$ into itself and

$$M = \{ m_f \mid f \in S^s$$

under the mapping $f \to m_f$. $M$ is actually a semigroup if we define multiplication on $M$ by

$$m_f m_g(u,v) = \sum_{s \in S} m_f(u,s) m_g(s,v) .$$

This semigroup is clearly isomorphic to the semigroup $S^s$ under composition of mappings.

With the above considerations, we can prove the following:

*Theorem.* Every semigroup may be embedded in a ring.

*Proof.* Let $G$ be a semigroup. It is isomorphic to a semigroup of mappings $G_x$ on a set $S$, i.e., a subsemigroup of $S^s$, hence a subsemigroup of $M$ [1, p. 20].

If we define $+$ and $\cdot$ on $Z_2^{S \times S}$ by $(i+j)(u,v) = i(u,v) + j(u,v)$,

$$(i \cdot j)(u,v) = \sum_{s \in S} i(u,s) j(s,v) .$$

This clearly makes $Z_2^{S \times S}$ a ring, and $M$ is a subsemigroup of its multiplicative semigroup.

## REFERENCES

1. E.S. Liapin, "Semigroups," *A.M.S. Translations of Mathematical Monographs,* Vol. 3, 1963.
2. E.S. Liapin, "Representations of Semigroups by Partial Mappings," *A.M.S. Transl.* (2) 27 (1963), pp. 289–296.

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