ON NON–BASIC TRIPLES

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Definition 1. A set of integers $\{b_i\}_i \geqslant 1$ will be called a base for the set of all integers whenever every integer n can be expressed uniquely in the form

$$n = \sum_{i=1}^{\infty} a_i b_i ,$$

where $a_i = 0$ or 1 and

$$\sum_{i=1}^{\infty} a_i < \infty .$$

Thus, a base is obtained by taking $b_i = \pm 2^i$ for each i so long as terms of each sign are used infinitely often. Also, a sequence $\{d_i\}_i \geqslant 1$ of odd numbers will be called basic whenever the sequence

$$\{d_i 2^{i-1}\} i \geqslant 1$$

is a base. If the sequence $\{d_i\}_i \geqslant 1$ of odd integers is such that $d_{i+s} = d_i$ for all i's, then the sequence is said to be periodic mod s and is denoted by $\{d_1, d_2, d_3, \cdots, d_s\}$.

Theorem 1. A basic sequence remains basic whenever a finite number of odd numbers is added, omitted, or replaced by other odd numbers.

Proof. This is proved in [1].

Theorem 2. A necessary and sufficient condition for the sequence $\{d_i\}_i \geqslant 1$ of odd integers, which is periodic mod s, to be basic is that

$$0 \neq \sum_{i=1}^{m} a_i 2^{i-1} d_i \equiv 0 \pmod{2^{ns} - 1}$$

is impossible for $n \geqslant 1$, and $a_i = 0$ or 1 for all $i \geqslant 1$.

Proof. This is also proved in [1].

Theorem 3. Let a, b, c be a periodic mod 3. If $a = d(2^{3K} + 1)$, where d is an integer and

(1) $$d + 2b + 4c \equiv 0 \pmod 7,$$

or

(2) $$b + 2d + 4c \equiv 0 \pmod 7,$$

or

(3) $$c + 2d + 4b \equiv 0 \pmod 7,$$

or

(4) $$c + 2b + 4d \equiv 0 \pmod 7,$$

or

(5) $$d + 2c + 4b \equiv 0 \pmod 7,$$

or

(6) $$b + 2c + 4d \equiv 0 \pmod 7,$$

then a,b,c is non-basic.

Proof. In case (1) holds, consider the expression

$$u = a + 2b + 2^2c + \cdots + 2^{3K-3}a + 2^{3K-2}b + 2^{3K-1}c + 2^{3K+1}b + 2^{3K+2}c + \cdots + 2^{6K-2}b + 2^{6K-1}c$$

$$= a(1 + 2^3 + \cdots + 2^{3K-3}) + 2b(1 + 2^3 + \cdots + 2^{6K-3}) + 2^2c(1 + 2^3 + \cdots + 2^{6K-3})$$

$$= a \cdot \frac{2^{3K-1}}{2^3 - 1} + 2b \cdot \frac{2^{6K-} - 1}{2^3 - 1} + 2^2c \cdot \frac{2^{6K} - 1}{2^3 - 1}$$

$$= d(2^{3K} + 1) \cdot \frac{2^{3K} - 1}{2^3 - 1} + 2b \cdot \frac{2^{6k} - 1}{2^3 - 1} + 2^2c \cdot \frac{2^{6k} - 1}{2^3 - 1} = \frac{(d + 2b + 2^2c)(2^{6K} - 1)}{2^3 - 1} .$$

It follows that u is divisible by $2^{6K} - 1$ since, by hypothesis,

$$(2^3 - 1) \mid (d + 2b + 2^2c).$$

Hence, by applying Theorem 2 with $n = 3$ and $s = 2k$, $\{a,b,c\}$ is not basic.

Suppose now that (2) holds and that $\{a,b,c\}$ is basic. By Theorem 1, we may interchange a with b the first $3K$ times these numbers appear in the sequence $\{a,b,c\}$ and still have a basic sequence. Consider

$$v = b + 2a + 2^2c + \cdots + 2^{3K-3}b + 2^{3K-2}a + 2^{3K-1}c + 2^{3K}b + 2^{3K+2}c + \cdots + 2^{6K-3}b + 2^{6K-1}c$$

$$= b(1 + 2^3 + \cdots + 2^{6K-3}) + 2a(1 + 2^3 + \cdots + 2^{3K-3}) + 2^2c(1 + 2^3 + \cdots + 2^{6K-3}).$$

As above, this reduces to

$$v = \frac{(b + 2d + 2^2c)(2^{6K} - 1)}{2^3 - 1}$$

and since $(2^3 - 1) \mid (b + 2d + 2^2c)$, v is divisible by $2^{6K} - 1$. But then, as before $\{a,b,c\}$ is not basic.

The remaining cases are handled in the same way, with an appropriate permutation of the first few terms in the sequence $\{a,b,c\}$ and so the proof is complete.

Theorem 4. Let

$$a = \frac{e(2^{6K} - 1)}{2^{2K} - 1} \qquad \text{and} \qquad b = \frac{d(2^{6K} - 1)}{2^{3K} - 1} ,$$

where e and d are integers, $K \neq 0$, and $3/K$. If $e + 2d + 2^2c$ is divisible by 7, then $\{a,b,c\}$ is non-basic.

Proof. Consider the expression

$$w = a + 2b + 2^2c + \cdots + 2^{2K-3}a + 2^{2K-2}b + 2^{2K-1}c + 2^{2K+1}b + 2^{2K+2}c + \cdots + 2^{3K-2}b + 2^{3K-1}c + \cdots + 2^{6K-1}c$$

$$= a(1 + 2^3 + \cdots + 2^{2K-3}) + 2b(1 + 2^3 + \cdots + 2^{3K-3}) + 2^2c(1 + 2^3 + \cdots + 2^{6K-3})$$

$$= a \cdot \frac{(2^{2K} - 1)}{2^3 - 1} + 2b \cdot \frac{(2^{3K} - 1)}{2^3 - 1} + 2^2c \cdot \frac{(2^{6K} - 1)}{2^3 - 1}$$

$$= e \cdot \frac{(2^{6K} - 1)}{2^{2K} - 1} \cdot \frac{(2^{2K} - 1)}{2^3 - 1} + 2d \cdot \frac{(2^{6K} - 1)}{2^{3K} - 1} \cdot \frac{(2^{3K} - 1)}{2^3 - 1} + 2^2c \cdot \frac{(2^{6K} - 1)}{2^3 - 1} = \frac{(e + 2d + 2^2c)(2^{6K} - 1)}{2^3 - 1} .$$

Since $e + 2d + 2^2c$ is divisible by 7, w is divisible by $2^{6K} - 1$, and $\{a,b,c\}$ is non-basic by Theorem 2.

Theorem 5. Let

$$a = e \cdot \frac{(2^{6K} - 1)}{2^{3K} - 1} \qquad \text{and} \qquad b = d \cdot \frac{(2^{6K} - 1)}{2^{3K} - 1} ,$$

where e and d are integers, $K \neq 0$, $3/K$. If

$$e + 2d + 2^2c$$

is divisible by 7, then $\{a,b,c\}$ is non-basic.

Proof. This time we set

$$\begin{aligned} v &= a + 2b + 2^2c + \cdots + 2^{3K-3}a + 2^{3K-2}b + 2^{3K-1}c + 2^{3K+2}c + \cdots + 2^{6K-1}c \\ &= a(1 + 2^3 + \cdots + 2^{3K-3}) + 2b(1 + 2^3 + \cdots + 2^{3K-3}) + 2^2c(1 + 2^3 + \cdots + 2^{6K-3}) \\ &= a \cdot \frac{2^{3K}-1}{2^3-1} + 2b \cdot \frac{2^{3K}-1}{2^3-1} + 2^2c \cdot \frac{2^{6K}-1}{2^3-1} \\ &= e \cdot \frac{2^{6K}-1}{2^{3K}-1} \cdot \frac{2^{3K}-1}{2^3-1} + 2d \cdot \frac{2^{6K}-1}{2^{3K}-1} \cdot \frac{2^{3K}-1}{2^3-1} + 2^2c \cdot \frac{2^{6K}-1}{2^3-1} \\ &= \frac{(e + 2d + 2^2c)(2^{6K}-1)}{2^3-1} . \end{aligned}$$

Since

$$e + 2d + 2^2c$$

is divisible by 7, v is divisible by $2^{6K} - 1$ and as before $\{a,b,c\}$ is non-basic. In a similar way, we obtain the following theorem.

Theorem 6. Let

$$a = \frac{e(2^{6K}-1)}{2^{2K}-1} \qquad \text{and} \qquad b = \frac{d(2^{6K}-1)}{2^{2K}-1} ,$$

where e and d are integers, $K \neq 0,\ 3/k$. If

$$e + 2d + 2^2c$$

is divisible by 7, then $\{a,b,c\}$ is non-basic.

Other similar interesting results may be found in another article in [2].

REFERENCES

1. N.G. deBruijn, "On Bases for the Set of Integers," *Publ. Math.,* Debrecen, 1 (1950), pp. 232–242.
2. C.T. Long and N. Woo, "On Bases for the Set of Integers," *Duke Math. Journal,* Vol. 38, No. 3, Sept. 1971, pp. 583–590.

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