SUMS AND PRODUCTS FOR RECURRING SEQUENCES

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In [1], we find many well known formulas which involve the sums of Fibonacci and Lucas numbers. For example, we have

(1) $$\sum_{i=1}^{n} F_i = F_{n+2} - 1, \qquad n \geq 1;$$

(2) $$\sum_{i=1}^{n} L_i = L_{n+2} - 3, \qquad n \geq 1;$$

(3) $$\sum_{i=1}^{n} F_{2i-1} = F_{2n}, \qquad n \geq 1;$$

(4) $$\sum_{i=1}^{n} L_{2i-1} = L_{2n} - 2, \qquad n \geq 1.$$

Hence, it is natural to ask if there exist summation formulas for other lists of Fibonacci and Lucas numbers. If such formulas exist it is then natural to ask if the formulas can be extended to other recurring sequences. The purpose of this paper is to show that both of these questions can be answered in the affirmative. To do this, we first recall the following [1, p. 59]

(5) $$F_{n+k} + F_{n-k} = F_n L_k, \quad k \text{ even};$$

(6) $$F_{n+k} + F_{n-k} = L_n F_k, \quad k \text{ odd};$$

(7) $$F_{n+k} - F_{n-k} = F_n L_k, \quad k \text{ odd};$$

(8) $$F_{n+k} - F_{n-k} = L_n F_k, \quad k \text{ even}.$$

Using $L_n = \alpha^n + \beta^n$ where α and β are the roots of $x^2 - x - 1 = 0$ with $\alpha = (1 + \sqrt{5})/2$, $\beta = (1 - \sqrt{5})/2$ it is easy to show that

(9) $$L_{n+k} + L_{n-k} = L_n L_k, \quad k \text{ even};$$

(10) $$L_{n+k} + L_{n-k} = 5F_n F_k, \quad k \text{ odd};$$

(11) $$L_{n+k} - L_{n-k} = L_n L_k, \quad k \text{ odd};$$

(12) $$L_{n+k} - L_{n-k} = 5F_n F_k, \quad k \text{ even}.$$

Observing that a sum involving 2^p terms, by combining pairs, reduces to a sum of 2^{p-1} terms, we were able to show

Theorem 1. If $k \geq 1$ then

(13) $$\sum_{i=0}^{2^j-1} F_{n+4ki} = F_{n+(2^j-1)2k} \prod_{i=1}^{j} L_{2^i k}.$$

Proof. If $j = 1$ then

$$\sum_{i=0}^{1} F_{n+4ki} = F_n + F_{n+4k} = L_{2k}F_{n+2k} = F_{n+(2^1-1)2k} \prod_{i=1}^{1} L_{2^ik}$$

and the theorem is true.

Assume the proposition is true for j. Using (5), we have

$$\sum_{i=0}^{2^{j+1}-1} F_{n+4ki} = L_{2k} \sum_{i=0}^{2^j-1} F_{n+2k+8ki}$$

$$= L_{2k}F_{n+2k+(2^j-1)4k} \prod_{i=1}^{j} L_{2^{i+1}k}$$

$$= F_{n+(2^{j+1}-1)2k} \prod_{i=1}^{j+1} L_{2^ik}$$

and the theorem is proved.

Using (9) and an argument like that of Theorem 1, we have

(14) $$\sum_{i=0}^{2^j-1} L_{n+4ki} = L_{n+(2^j-1)2k} \prod_{i=1}^{j} L_{2^ik}, \quad k \geq 1.$$

Using (8) and (14) with $j - 1$ in place of j, $n + 2k$ in place of n and $2k$ in place of k, one has

(15) $$\sum_{i=0}^{2^j-1} (-1)^{i+1} F_{n+4ki} = F_{2k}L_{n+(2^j-1)2k} \prod_{i=2}^{j} L_{2^ik}, \quad k \geq 1.$$

Similarly, with the aid of (12) and Theorem 1, one obtains

(16) $$\sum_{i=0}^{2^j-1} (-1)^{i+1} L_{n+4ki} = 5F_{2k}F_{n+(2^j-1)2k} \prod_{i=2}^{j} L_{2^ik}, \quad k \geq 1.$$

From (9) and (14), we have

(17) $$\sum_{i=0}^{2^j-1} L_{n+(2i-1)k} = L_{n+(2^{j-1}-1)2k} \prod_{i=0}^{j-1} L_{2^ik}, \quad k \text{ even}$$

while Theorem 1 with the aid of (12) gives

(18) $$\sum_{i=0}^{2^j-1} (-1)^{i+1} L_{n+(2i-1)k} = 5F_kF_{n+(2^{j-1}-1)2k} \prod_{i=1}^{j-1} L_{2^ik}, \quad k \text{ even}.$$

Theorem 1 together with (5) can be used to show

(19) $$\sum_{i=0}^{2^j-1} F_{n+(2i-1)k} = F_{n+(2^{j-1}-1)2k} \prod_{i=0}^{j-1} L_{2^ik}, \quad k \text{ even}$$

while (8) with (14) yields

(20) $$\sum_{i=0}^{2^j-1} (-1)^{i+1} F_{n+(2i-1)k} = F_kL_{n+(2^{j-1}-1)2k} \prod_{i=1}^{j-1} L_{2^ik}, \quad k \text{ even}.$$

Since we have used (5) and (8) as well as (9) and (12) on several occasions, it seems natural to ask if formulas exist using (6) and (7) as well as (10) and (11). With this in mind, we developed the next four formulas.

By use of (10) and (11), respectively with Theorem 1, we have

(21) $$\sum_{i=0}^{2^j-1} L_{n+(2i-1)k} = 5F_kF_{n+(2^{j-1}-1)2k} \prod_{i=1}^{j-1} L_{2^ik}, \quad k \text{ odd}$$

and

(22) $$\sum_{i=0}^{2^j-1} (-1)^{i+1} F_{n+(2i-1)k} = F_{n+(2^{j-1}-1)2k} \prod_{i=0}^{j-1} L_{2^ik}, \quad k \text{ odd}.$$

Finally, if we apply (6) and (7) respectively with (14) we are able to show that

(23) $$\sum_{i=0}^{2^j-1} F_{n+(2i-1)k} = F_kL_{n+(2^{j-1}-1)2k} \prod_{i=1}^{j-1} L_{2^ik}, \quad k \text{ odd}$$

and

(24) $$\sum_{i=0}^{2^j-1} (-1)^{i+1} L_{n+(2i-1)k} = L_{n+(2^{j-1}-1)2k} \prod_{i=0}^{j-1} L_{2^ik}, \quad k \text{ odd}.$$

To lift the results above to the generalized Fibonacci sequence which is defined recursively by

(25) $$H_0 = q, \quad H_1 = p, \quad H_n = H_{n-1} + H_{n-2}, \quad n \geqslant 2$$

it is necessary and sufficient to examine formulas comparable to (5) through (12). To do this, we first define a generalized Lucas sequence by

(26) $$G_n = H_{n+1} + H_{n-1}.$$

In Horadam [3], it is shown that

(27) $$H_n = (r\alpha^n - s\beta^n)/2\sqrt{5},$$

where $r = 2(p - q\beta)$, $s = 2(p - q\alpha)$ and α, β are the usual roots of $x^2 - x - 1 = 0$. Furthermore, he shows that

(28) $$H_{n+k} = H_{n-1}F_k + H_nF_{k+1},$$

where the F_k are the Fibonacci numbers.

Using (27) and Binet's formula for F_k, a straightforward argument shows that

(29) $$H_nF_{k-1} - H_{n-1}F_k = (-1)^kH_{n-k}.$$

By (28) and (29) with the aid of $L_k = F_{k+1} + F_{k-1}$, we have

(30) $$H_{n+k} + H_{n-k} = H_nL_k, \quad k \text{ even}$$

and

(31) $$H_{n+k} - H_{n-k} = H_nL_k, \quad k \text{ odd}.$$

If we use (25), (28), and (29) together with the fact that $F_k = F_{k+1} - F_{k-1}$, we have

(32) $$H_{n+k} + H_{n-k} = G_nF_k, \quad k \text{ odd}$$

and

(33) $$H_{n+k} - H_{n-k} = G_nF_k, \quad k \text{ even}.$$

Replacing n by $n + k$ in (26) and using (28), we have

(34) $$G_{n+k} = H_{n-1}L_k + H_nL_{k+1}$$

while replacing n by $n - k$ in (26) and applying (29) gives

(35) $$G_{n-k} = (-1)^k(H_{n-1}L_k - H_nL_{k-1}).$$

Applying (34) and (35) as we did (28) and (29), we obtain

(36) $$G_{n+k} + G_{n-k} = G_nL_k, \quad k \text{ even};$$

(37) $$G_{n+k} + G_{n-k} = 5H_nF_k, \quad k \text{ odd};$$

(38) $$G_{n+k} - G_{n-k} = G_n L_k, \quad k \text{ odd};$$

(39) $$G_{n+k} - G_{n-k} = 5H_n F_k, \quad k \text{ even}.$$

Examining (30) through (33) and (36) through (39) with H replaced by F and G replaced by L, we obtain properties (5) thorugh (12). Hence, it is clear that identities (13) through (24) can be lifted to the generalized Fibonacci and Lucas sequences and in fact are

(40) $$\sum_{i=0}^{2^j-1} H_{n+4ki} = H_{n+(2^j-1)2k} \prod_{i=1}^{j} L_{2^ik}, \quad k \geqslant 1;$$

(41) $$\sum_{i=0}^{2^j-1} G_{n+4ki} = G_{n+(2^j-1)2k} \prod_{i=1}^{j} L_{2^ik}, \quad k \geqslant 1;$$

(42) $$\sum_{i=0}^{2^j-1} (-1)^{i+1} H_{n+4ki} = F_{2k} G_{n+(2^j-1)2k} \prod_{i=2}^{j} L_{2^ik}, \quad k \geqslant 1;$$

(43) $$\sum_{i=0}^{2^j-1} (-1)^{i+1} G_{n+4ki} = 5F_{2k} H_{n+(2^j-1)2k} \prod_{i=2}^{j} L_{2^ik}, \quad k \geqslant 1;$$

(44) $$\sum_{i=0}^{2^j-1} G_{n+(2i-1)k} = G_{n+(2^{j-1}-1)2k} \prod_{i=0}^{j-1} L_{2^ik}, \quad k \text{ even};$$

(45) $$\sum_{i=0}^{2^j-1} (-1)^{i+1} G_{n+(2i-1)k} = 5F_k H_{n+(2^{j-1}-1)2k} \prod_{i=1}^{j-1} L_{2^ik}, \quad k \text{ even};$$

(46) $$\sum_{i=0}^{2^j-1} H_{n+(2i-1)k} = H_{n+(2^{j-1}-1)2k} \prod_{i=0}^{j-1} L_{2^ik}, \quad k \text{ even};$$

(47) $$\sum_{i=0}^{2^j-1} (-1)^{i+1} H_{n+(2i-1)k} = F_k G_{n+(2^{j-1}-1)2k} \prod_{i=1}^{j-1} L_{2^ik}, \quad k \text{ even};$$

(48) $$\sum_{i=0}^{2^j-1} G_{n+(2i-1)k} = 5F_k H_{n+(2^{j-1}-1)2k} \prod_{i=1}^{j-1} L_{2^ik}, \quad k \text{ odd};$$

(49) $$\sum_{i=0}^{2^j-1} (-1)^{i+1} H_{n+(2i-1)k} = H_{n+(2^{j-1}-1)2k} \prod_{i=0}^{j-1} L_{2^ik}, \quad k \text{ odd};$$

(50) $$\sum_{i=0}^{2^j-1} H_{n+(2i-1)k} = F_k G_{n+(2^{j-1}-1)2k} \prod_{i=1}^{j-1} L_{2^ik}, \quad k \text{ odd};$$

(51) $$\sum_{i=0}^{2^j-1} (-1)^{i+1} G_{n+(2i-1)k} = G_{n+(2^{j-1}-1)2k} \prod_{i=0}^{j-1} L_{2^ik}, \quad k \text{ odd}.$$

The infinite sequence $\{x_n\}_{n=1}^{\infty}$ is called a recurring sequence if, from a certain point on, every term can be represented as a linear combination of the preceding terms of the sequence. Hence, the sequence $\{U_n(x,y)\}_{n=1}^{\infty}$

defined recursively by

(52) $$U_0(x,y) = 0, \quad U_1(x,y) = 1, \quad U_n(x,y) = xU_{n-1}(x,y) + yU_{n-2}(x,y), \quad n \geq 2.$$

where $U_n(x,y) \in F[x,y]$, F any field is a recurring sequence.

If we let λ_1 and λ_2 be the roots of the equation $\lambda^2 - x\lambda - y = 0$, where we assume $\lambda_1 = (x + \sqrt{x^2 + 4y})/2$, $y \neq 0$, and $x^2 + 4y$ is a nonperfect square different from zero, then it is easy to show that

(53) $$U_n(x,y) = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}.$$

Furthermore, if we let

(54) $$V_n(x,y) = \lambda_1^n + \lambda_2^n$$

then

(55) $$V_n(x,y) = yU_{n-1}(x,y + U_{n+1}(x,y).$$

Because of the y coefficient, the formulas (5) through (12) do not follow the same pattern for this recurring sequence. However, it can be shown using (53) through (55) together with the facts $\lambda_1\lambda_2 = -y$ and $\lambda_1 + \lambda_2 = x$ that

(56) $$U_{n+k}(x,y) + y^k U_{n-k}(x,y) = U_n(x,y)V_k(x,y), \quad k \text{ even};$$

(57) $$U_{n+k}(x,y) + y^k U_{n-k}(x,y) = V_n(x,y)U_k(x,y), \quad k \text{ odd};$$

(58) $$U_{n+k}(x,y) - y^k U_{n-k}(x,y) = U_n(x,y)V_k(x,y), \quad k \text{ odd};$$

(59) $$U_{n+k}(x,y) - y^k U_{n-k}(x,y) = V_n(x,y)U_k(x,y), \quad k \text{ even}$$

(60) $$V_{n+k}(x,y) + y^k V_{n-k}(x,y) = V_n(x,y)V_k(x,y), \quad k \text{ even}$$

(61) $$V_{n+k}(x,y) + y^k V_{n-k}(x,y) = (x^2 + 4y)U_n(x,y)U_k(x,y), \quad k \text{ odd};$$

(62) $$V_{n+k}(x,y) - y^k V_{n-k}(x,y) = V_n(x,y)V_k(x,y), \quad k \text{ odd};$$

(63) $$V_{n+k}(x,y) - y^k V_{n-k}(x,y) = (x^2 + 4y)U_n(x,y)U_k(x,y), \quad k \text{ even}.$$

Because of the y^k, it is quite obvious that formulas (13) through (24) do not have the same form for the recurring sequences $\{U_n(x,y)\}$ and $\{V_n(x,y)\}$. If we let the coefficients of $U_{n-2}(x,y)$ in (52) be $y = 1$ then the sequences $\{U_n(x,y)\}$ and $\{V_n(x,y)\}$ are sequences of polynomials in x. In fact, they are respectively the sequences of Fibonacci and Lucas polynomials. With $y = 1$, it is easy to see that formulas (56) through (63) are of the same nature as (5) through (12) with F in place of U and L in place of V. Hence, the formulas (13) through (24) can be lifted to the sequences $\{U_n(x,y)\}$ and $\{V_n(x,y)\}$ if $y = 1$ by replacing F_n by $U_n(x,1)$ and L_n by $V_n(x,1)$. Of course, we have $x^2 + 4$ in palce of 5 in formulas (16), (18), and (21).

In conclusion, we will examine whiat happens if we consider the recurring sequence $\{H_n(x,y)\}_{n=1}^{\infty}$ where

(64) $$H_0(x,y) = f(x,y), \quad H_1(x,y) = g(x,y),$$
$$H_n(x,y) = xH_{n-1}(x,y) + yH_{n-2}(x,y), \quad n \geq 2.$$

By using properties of difference equations, it is easy to show that

(65) $$H_n(x,y) = (r(x,y)\lambda_1^n - s(x,y)\lambda_2^n)/2\sqrt{x^2 + 4y}$$

where λ_1 and λ_2 are as before, $r(x,y) = 2(g(x,y) - f(x,y)\lambda_2)$, and $s(x,y) = 2(g(x,y) - f(x,y)\lambda_1)$.

If we let

(66) $$G_n(x,y) = (r(x,y)\lambda_1^n + s(x,y)\lambda_2^n)/2$$

then

(67) $$G_n(x,y) = yH_{n-1}(x,y) + H_{n+1}(x,y).$$

Using (53) and (65), a direct calculation will show that

(68) $$H_n(x,y)U_{k+1}(x,y) + yH_{n-1}(x,y)U_k(x,y) = H_{n+k}(x,y)$$

and

(69) $$H_n(x,y)U_{k-1}(x,y) - H_{n-1}(x,y)U_k(x,y) = (-1)^k y^{k-1} H_{n-k}(x,y).$$

If we use (57) with (67) and (68) and remember that $U_1(x,y) = 1$, we obtain

(70) $$G_{n+k}(x,y) = yH_{n-1}(x,y)V_k(x,y) + H_n(x,y)V_{k+1}(x,y).$$

Using (55) with (69) and (67), it can be shown that

(71) $$H_{n-1}(x,y)V_k(x,y) - H_n(x,y)V_{k-1}(x,y) = (-1)^k y^{k-1} G_{n-k}(x,y).$$

Letting k be odd or even in (68) through (71), we have

(72) $$H_{n+k}(x,y) + y^k H_{n-k}(x,y) = H_n(x,y)V_k(x,y), \quad k \text{ even};$$

(73) $$H_{n+k}(x,y) + y^k H_{n-k}(x,y) = G_n(x,y)U_k(x,y), \quad k \text{ odd};$$

(74) $$H_{n+k}(x,y) - y^k H_{n-k}(x,y) = H_n(x,y)V_k(x,y), \quad k \text{ odd};$$

(75) $$H_{n+k}(x,y) - y^k H_{n-k}(x,y) = G_n(x,y)U_k(x,y), \quad k \text{ even};$$

(76) $$G_{n+k}(x,y) + y^k G_{n-k}(x,y) = G_n(x,y)V_k(x,y), \quad k \text{ even};$$

(77) $$G_{n+k}(x,y) + y^k G_{n-k}(x,y) = (x^2+4y)H_n(x,y)U_k(x,y), \quad k \text{ odd};$$

(78) $$G_{n+k}(x,y) - y^k G_{n-k}(x,y) = G_n(x,y)V_k(x,y), \quad k \text{ odd};$$

(79) $$G_{n+k}(x,y) - y^k G_{n-k}(x,y) = (x^2+4y)H_n(x,y)U_k(x,y), \quad k \text{ even}.$$

Observe that if we replace H by U and G by V then Eqs. (72) through (79) yield Eqs. (56) through (63).

If we let $y = 1$ in (64) then Eqs. (72) through (79) are those of (30) through (33) and (36) through (39) where we replace $V_n(x,y)$ by L_n, $H_n(x,y)$ by H_n, $G_n(x,y)$ by G_n, and $U_n(x,y)$ by F_n. The same substitutions in (40) through (51) will give us the summation-product relations relative to the sequences $\{H_n(x,y)\}$ and $\{G_n(x,y)\}$ if $y = 1$.

In conclusion, we observe several other results which are a direct consequence of the formulas of this paper [2; p. 19].

If we replace n by $k+1$ in (5) through (8) we have F_k, L_k, F_{k+1}, and L_{k+1} are relatively prime to F_{2k+1} for $k \geq 1$. If we let $n = k+2$ in (5) through (8), we have F_k, L_k, F_{k+2}, and L_{k+2} are all relatively prime to F_{2k+2} for $k \geq 1$. Letting $n = k+1$ in (9) through (12), we see that F_k, L_k, F_{k+1}, and L_{k+1} are all relatively prime to L_{2k+1}.

If we let $n = k+1$ in (56) through (59) with $y = 1$ we see that the Fibonacci polynomials $U_{2k+1}(x,1) \pm 1$ are factorable for $k \geq 2$. If $n = k$ with $y = 1$ in (56) through (59) then $U_{2k}(x,1)$ is factorable for $k \geq 2$.

REFERENCES

1. V.E. Hoggatt, Jr., *Fibonacci and Lucas Numbers,* Houghton Mifflin Co., 1969.
2. V.E. Hoggatt, Jr., and G.E. Bergum, "Divisibility and Congruence Relations," *The Fibonacci Quarterly,* Vol. 12, No. 2 (April 1974), pp. 189–195.
3. A.F. Horadam, "A Generalized Fibonacci Sequence," *American Mathematical Monthly,* Vol. 66, 1959, pp. 445–459.
4. I. Niven and H. Zuckerman, *Introduction to the Theory of Numbers,* John Wiley and Sons, Inc., 1960.

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[Continued from Page 110.]

(3B) *If* k *is an integer for which Fermat's Last Theorem is true, then there is no pythagorean triangle with the hypotenuse and one of the legs equal to* k^{th} *powers of natural numbers.*

Proofs of 1B and 2B are provided in the complete text, but 3B remains an open question.

The authors have attempted to compile a complete bibliography related to pythagorean triangles. Included in the bibliography are 111 references to journal articles, 66 references to problems (with solutions) in *Amer. Math Monthly,* 17 references to notes in *Math. Gaz.,* and 12 references to notes in *Math. Mag.* Since it is impossible to compile such a bibliography without some omissions, the authors would appreciate receiving any references not already included in the bibliography.

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