IDIOT'S ROULETTE REVISITED

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The winter 1973 issue of the California Mathematics Council *Bulletin* carried an article under the title "Idiot's Roulette." It discussed a counting-out puzzle, in which N people stand in a circle surrounding an executioner, who goes around and around the circle, shooting every second person as he counts. The problem is to determine the "safe" position, X, as a function of N. That is—which will be the last person left, according to the original numbering? An intuitive solution was presented, developed by looking for patterns, and the author asked for further comments on possible proofs.

The problem is a special case of a more general counting-out problem I had been playing with the previous fall, although in a somewhat less bloodthirsty fashion, —and the analysis which provides an iterative solution for the general case incidentally yields a closed-form solution for the special case where the countoff spacing = 2.

The general problem: Given N places around the circle and a countoff spacing = C, such that every C^{th} place drops out, the count continuing around the circle until only one place is left, —which of the numbers 1 to N will be the last place L ?

Assume the count "1" starts with place number 1. A different starting point simply rotates the problem around the circle, changing nothing essential. This seemingly trivial observation, however, provides a key to the analysis and solution of the problem. So let us consider what happens if we start the count at some other number, say at $J + 1$ instead of 1. This is equivalent to rotating the problem J places around the circle, so the game would end at $L + J$ instead of L, unless $L + J > N$, in which case the modular nature of our numbering makes the last place $L + J - N$.

Now return to the original problem. The count starts at place number 1, with countoff spacing C and N people. Call the solution for the last place winner L_N. (For simplicity in the following discussion, we shall restrict ourselves to the case where $N \geqslant C - 2$. See footnote 1 for more complete analysis.) L_N is a function of C and N. Now consider the problem for the same countoff spacing C, but with one more person in the circle. After our first loser is counted out at place C, this reduces to a circle of N places in which the count starts at $C + 1$. So $L_{N+1} = L_N + C$, unless $L_N + C > N + 1$, in which case we have

$$L_{N+1} = L_N + C - (N + 1).$$

The table on the following page shows the situation, for example, for $C = 2$, and several values of N.

I shall now introduce some terminology which will help us develop an iterative solution for the general problem. Noting that, for a given C, each time we add a place to the circle we add C to the old solution, write the solution in the form $L_N = CN - I_N$, since some integer I_N certainly must exist which will make the statement true. (Example: In the table, where $C = 2$ and $N = 4$, $L_4 = 2(4) - 7$, and $I_4 = 7$. For $N = 5, 6,$ and 7 also, $I_N = 7$. For $N = 8$, however, this is no longer true. $I_8 = 15$.)

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[1] If $C = 1$, the problem is trivial, with $L_N = N$ for all N. If $C > 1$, the general statement becomes:

$$L_{N+1} = L_N + C - T_N(N+1) \quad \text{and} \quad I_{N+1} = I_N + T_N(N+1), \quad \text{where} \quad T_N = \left[C - \frac{I_N + 1}{N + 1} \right] .$$

For $N \geqslant C - 2$, T_N must be either 0 or 1, and the analysis in the article holds completely. For small N, however, some of the S values generated may not actually be used, with the general statement being: If $I_N = S_K$,

$$I_{N+1} = S_{k+T_N} .$$

For example, if $C = 4$, $S = \{3, 5, 7, 10, 14, 19, 26, \cdots\}$. $I_1 = S_1 = 3$; $I_2 = S_3 = 7$, since

$$T_N = \left[4 - \frac{3+1}{2} \right] = 2 .$$

Similarly, for $C = 7$, $S = \{6, 8, 10, 12, 15, 18, 22, 26, 31, 37, \cdots\}$ $I_1 = S_1 = 6$; $I_2 = S_4 = 12$; $I_3 = S_6 = 18$; $I_4 = S_8 = 26$; $I_5 = S_9 = 31$.

N	L_N	L_{N+1}	$(C = 2)$
4	1	3	
5	3	5	
6	5	7	
7	7	1	(7 + 2 – 8)
8	1	3	
9	3	5	

The problem now is to find the appropriate I_N such that $L_N = CN - I_N$, where $1 \le L_N \le N$.

We can restate the condition that $CN - I_N \le N$ to obtain

(1) $$N \le \frac{I_N}{C-1}\ .$$

Next look at the statements about $L_{N+1} = C(N+1) - I_{N+1}$: If

(2) $$L_N + C \le N+1, \qquad L_{N+1} = L_N + C = CN - I_N + C = C(N+1) - I_N\ .$$

Thus, if $L_N + C \le N+1$, $I_{N+1} = I_N$, while if

(3) $$L_N + C > N+1, \qquad L_{N+1} = L_N + C - (N+1) = C(N+1) - (I_N + N + 1) \quad \text{and} \quad I_{N+1} = I_N + N + 1\ .$$

Call S the set of *distinct* subtraction integers, where $I_N \ne I_{N+1}$, and let M be the set of $(N+1)$ values at which this occurs. Then we can restate, from (3), $S_{k+1} = S_k + M_k$; and also rewrite the inequality

$$L_N + C = CN - S_k + C = C(N+1) - S_k = CM_k - S_k > M_K\ ,$$

from which we obtain:

(4) $$M_k > \frac{S_k}{C-1}\ .$$

Similarly, rewriting (1) we have

$$M_k - 1 \le \frac{S_k}{C-1}\ , \qquad \text{or} \qquad M_k \le \frac{S_k}{C-1} + 1\ .$$

Combining this statement with (4), we obtain

(5) $$\frac{S_k}{C-1} < M_k \le \frac{S_k}{C-1} + 1\ ,$$

which can be solved in terms of the greatest integer function:

(6) $$M_k = \left[\frac{S_k}{C-1}\right] + 1,$$

(where $[x]$ is defined as the greatest integer $\le x$).

For a circle where $N = M_k$ places, then, our last place winner

$$L = CM_k - (S_k + M_k).$$

Since $S_{k+1} = S_k + M_k$, we have the following iterative formula for subtraction integers:

(7) $$S_{k+1} = S_k + \left[\frac{S_k}{C-1}\right] + 1 = \left[\frac{C}{C-1}\,S_k\right] + 1\ .$$

To obtain a starting point for the set of S values, we note that, for $N = 1$, $L_N = 1$, whatever the value of C. Hence $1 = C - S_1$, and $S_1 = C - 1$. Given a particular C, we can generate a set of subtraction integers[1]. For example, for $C = 3$:

$$S_1 = 2; \quad S_{k+1} = \left[\frac{3}{2}\,S_k\right] + 1,$$

and the set of S values is

$$\{\, 2, 4, 7, 11, 17, 26, 40, 61, \cdots \}$$

To apply the formula $L_N = CN - S_k$, we simply choose the proper S_k so that[2]

$$1 \leqslant L_N \leqslant N.$$

(Uniqueness of S_k can be shown readily from the equivalent condition that $(C-1)N \leqslant S_k \leqslant CN$.)

For the very special case of $C = 2$, the solution reduces neatly to a closed form, because

$$\frac{C}{C-1} = 2,$$

an integer. We can show by mathematical induction that for $C = 2$,

$$S_k = 2^k - 1,$$

since

$$S_1 = C - 1 = 2^1 - 1,$$

and

$$S_{k+1} = 2S_k + 1 = 2(2^k - 1) + 1 = 2^{k+1} - 1.$$

Therefore we can write: If

(8) $$C = 2, \qquad L = 2N - (2^k - 1) \quad \text{and} \quad 1 \leqslant 2N - (2^k - 1) \leqslant N.$$

By rewriting the inequality in (8) we can obtain an explicit solution for k in terms of N. We have

$$2^k - 1 + 1 \leqslant 2N;$$

hence $2^k \leqslant 2N$, and $k \leqslant 1 + \log_2 N$. We Also have

$$2N \leqslant N + 2^k - 1;$$

therefore $N \leqslant 2^k - 1$, and $N < 2^k$. Thus $\log_2 N < k$. Combining the inequalities:

(9) $$\log_2 N < k \leqslant 1 + \log_2 N, \quad \text{so} \quad k = 1 + [\log_2 N].$$

An explicit formula can therefore be written for L.

(10) $$L = 2N - (2^{1+[\log_2 N]} - 1) = 1 + 2(N - 2^{[\log_2 N]})$$

and the roulette player can avoid the executioner if he quickly counts how many share his possible fate and uses his fingers to calculate powers of 2![3]

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[2]I tried a number of computer runs to obtain M_k and S_k sets for various values of C. The resulting sequences of numbers looked hauntingly familiar, as though they ought to be expressible in some more elegant form. It might be interesting to follow up on this.

[3]This paper also provides a solution for the Population Explosion problem of Brother Alfred Brousseau, *The Fibonacci Quarterly,* Vol. 6, No. 1 (February 1968), pp. 58–59.

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