which tells us that the C_n for even n lie on a line with slope $-x/(x^2+2)$. Further, since the products of the slopes is -1, these lines are perpendicular. This proves Theorem 8.

From the above result it follows almost trivially that

Theorem 9. If D_n is the distance of C_n from the point of intersection of the two lines of centers, then

$$D_n = \frac{\mathcal{L}_n^*(x)\sqrt{x^4+5x^2+4}}{2(x^2+4)} \quad ,$$

where the $\mathcal{L}_n^*(x)$ are the generalized Lucas polynomials

$$\mathcal{L}_1^* = p, \qquad \mathcal{L}_2^* = xp+2, \qquad \text{and} \qquad \mathcal{L}_{n+2}^*(x) = x\mathcal{L}_{n+1}^*(x) + \mathcal{L}_n^*(x).$$

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A LEAST INTEGER SEQUENCE INVESTIGATION

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In the fall semester of 1964, four students, Robert Lera, Ron Staszkow, Rod Arriaga, and Robert Martel began an investigation along with their teacher, Brother Alfred Brousseau, of a problem that arose in connection with a Putnam examination question. The problem was to prove that if

$$P_{n+1} = [p_n + p_{n-1} + p_{n-2}]/p_{n-3}$$

produced an endless sequence of integers while the quantities p_i remained less in absolute value than an upper bound A, then the sequence must be periodic. The divergent idea that led to the research was this: How can one insure an infinite sequence of integers from such a recursion formula? One quick answer was to use the greatest integer function.

Initially an investigation was begun on:

$$a_{n+1} = \left[\frac{a_n + a_{n-1}}{a_{n-2}} \right] \quad ,$$

where the square brackets mean: "take the greatest integer less than or equal to the quantity enclosed within the brackets." Very quickly, zero entered into the sequence with the result that there were mathematical complications once it arrived at the denominator.

To avoid this problem, it was decided to try using "the least integer function" instead of the greatest integer function. The notation adopted was:

$$[x]^* = n \, ,$$

where n is the least integer greater than or equal to x. With this approach starting with three positive integers the function:

$$a_{n+1} = \left[\frac{a_n + a_{n-1}}{a_{n-2}} \right]^*$$

gives terms that are always $\geqslant 1$.

The problem was enlarged by introducing two parameters, p and q, defining:

$$a_{n+1} = [(pa_n + qa_{n-1})/a_{n-2}]^*.$$

For any given pair (p,q) we have a set of sequences determined by assigning any three initial positive integers (a_1, a_2, a_3). We shall speak of these as the sequences belonging to (p,q).

Our least integer function representation is actually equivalent to two inequality relations. Thus if

$$[A]^* = B$$

the equivalent inequality statements are:

$$A \leqslant B \quad \text{and} \quad A > B - 1,$$

where B is an integer.

A GENERAL PERSPECTIVE

Early in the work with sequences, when examining such cases as (1,1), (1,2), (2,1), (2,2), (2,3), (3,4), it was noted that when $p \geqslant q$, only periodic sequences are found, whereas when $p < q$, there were also non-periodic sequences as well. This became one of the general topics of the research. One theorem along these lines can be stated immediately.

Theorem. Every set of sequences (p,q) has at least one periodic sequence.

Proof. Consider the sequence determined by the three quantities $p + q, p + q, p + q$. The fourth term calculates out as $p + q$. Thus this sequence will continue indefinitely with the single quantity $p + q$.

SEQUENCES OF TYPE (1,1)

The general recursion relation for this type of sequence is:

$$a_{n+1} = [(a_n + a_{n-1})/a_{n-2}]^*.$$

To explore this case it was found convenient to set up a table of the quantities a_1, a_2, a_3 for all values $a_i \leqslant 6$, $(i = 1,2,3)$. Starting with 1,1,1, for example, one obtains: 1,1,1,2,3,5,4,3,2,2,2,$\cdots$. This was called a B sequence, the period being of length one and consisting of the single number 2.

Four distinct periods were found:

$A \cdots$ 4,2,2,1,2,2,4,3 (Length 8) $B \cdots$ 2   (Length 1)
$C \cdots$ 1,3,1,4,2,6,2,4 (Length 8) $D \cdots$ 2,3 (Length 2) .

It was demonstrated that these are the only four periods among sequences of type (1,1), the argument following along these lines.

(1) Only a finite number of sequences all of whose elements are less than or equal to 6 lead to D.
(2) Any sequence involving an element 1 leads to one of the periods A, B, or C.
(3) For a sequence determined by three elements none of which is 1 and at least one of which is greater than 6, an element 1 will eventually appear in the sequence.

SEQUENCES OF TYPE (3,1)

Using starting numbers up to six, five periods were found:

$A:$ 1,1,1,4,13,43,36,12,2,1,1,2,7,23,38,20,5. (Length 17)
$B:$ 2,2,2,4,7,13,12,7,3,2,2,3,6,11,13,9,4. (Length 17)
$C:$ 3,3,3,4,5,7,7,6,4. (Length 9)
$D:$ 5,4. (Length 2)
$E:$ 4. (Length 1).

We arrived at no proof that these were all the periods in this case.

SEQUENCES OF PERIOD TWO IN (p,p), $(p, p + 1)$, and $(p + 1, p)$

An extensive investigation was made of the number of sequences of period two for these sets of sequences (21 pages). In all these cases it was possible to arrive at a formula of some complexity for determining the number of such sequences.

IMPOSSIBILITY OF VARIOUS PERIODS

Regardless of the values of p and q, it was shown by a detailed consideration of inequalities that it was impossible to have a period of length three. Similarly, it was proved that there are no periods of length four or five. With six, the work became quite complicated and as a result there was no demonstration of the impossibility of this case.