FIBONACCI AND RELATED SEQUENCES IN PERIODIC TRIDIAGONAL MATRICES

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1. INTRODUCTION

Tridiagonal matrices are matrices like

$$(1)\qquad \begin{bmatrix} b_1 & c_1 & 0 & 0 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & 0 & 0 \\ 0 & a_3 & b_3 & c_3 & 0 & 0 \\ 0 & 0 & a_4 & b_4 & c_4 & 0 \\ 0 & 0 & 0 & a_5 & b_5 & c_5 \\ 0 & 0 & 0 & 0 & a_6 & b_6 \end{bmatrix}$$

and are made up of three diagonal sequences $\{a_i\}, \{b_i\}, \{c_i\}$ of real or complex numbers. They are of much use in the numerical analysis of matrices. They also have interesting arithmetical properties being connected with the theories of continued fractions, recurring sequences of the second order, and, in special cases, permutations, graph theory, and partitions. We shall be considering two functions of such matrices, the determinant and the permanent.

By the permanent of the matrix

$$A = \{a_{ij}\}_{n\times n}$$

is meant the sum

$$\operatorname{per} A = \sum_{(\pi)} a_{1,\pi(1)}a_{2,\pi(2)}\cdots a_{n,\pi(n)}$$

extending over all permutations

$$\pi:\ \begin{pmatrix} 1, & 2, \cdots, & n \\ \pi(1), & \pi(2), \cdots, & \pi(n) \end{pmatrix} .$$

Thus the definition of the permanent is simpler than the corresponding definition of the determinant in that no distinction is made between odd and even permutations. In spite of this apparent simplicity, permanents are usually much more difficult than determinants in their computation and manipulation. For tridiagonal matrices, however, determinants and permanents are not very different. In fact we see that

$$\operatorname{per}\begin{bmatrix} b_1 & c_1 \\ a_2 & b_2 \end{bmatrix} = b_1b_2 + a_2c_1$$

and

$$\operatorname{per}\begin{bmatrix} b_1 & c_1 & 0 \\ a_2 & b_2 & c_2 \\ 0 & a_3 & b_3 \end{bmatrix} = b_1b_2b_3 + a_2b_3c_1 + a_3b_1c_2$$

and, in general, the permanent of the tridiagonal matrix based on $\{a_i\}, \{b_i\}, \{c_i\}$ is equal to the determinant of the matrix based on $\{-a_i\}, \{b_i\}, \{c_i\}$. Thus it is sufficient and simpler to consider the permanent function of tridiagonal matrices. In fact we shall need only the method of expansion by minors in developing what follows.

2. STANDARDIZATION OF TRIDIAGONAL MATRICES

For our present purposes we make the assumption that the elements b on the main diagonal are all different from zero. It is therefore possible to divide the elements in each row by its main diagonal element. Thus we obtain a matrix of the form

$$(2) \qquad \begin{bmatrix} 1 & C_1 & 0 & 0 & 0 & 0 \\ A_2 & 1 & C_2 & 0 & 0 & 0 \\ 0 & A_3 & 1 & C_3 & 0 & 0 \\ 0 & 0 & A_4 & 1 & C_4 & 0 \\ 0 & 0 & 0 & A_5 & 1 & C_5 \\ 0 & 0 & 0 & 0 & A_6 & 1 \end{bmatrix}$$

whose permanent (or determinant) is related to that of the original matrix (1) by the factor $b_1 b_2 \cdots b_6$. Our next step towards standardization is to observe that the permanent of (2) is not a function of A_2 *and* C_1 but only of their product A_2C_1. To see this, we expand the permanent by minors in the first column obtaining

$$\text{per} \begin{bmatrix} 1 & C_2 & 0 & 0 & 0 \\ A_3 & 1 & C_3 & 0 & 0 \\ 0 & A_4 & 1 & C_4 & 0 \\ 0 & 0 & A_5 & 1 & C_5 \\ 0 & 0 & 0 & A_6 & 1 \end{bmatrix} + A_2C_1 \,\text{per} \begin{bmatrix} 1 & C_3 & 0 & 0 \\ A_4 & 1 & C_4 & 0 \\ 0 & A_5 & 1 & C_5 \\ 0 & 0 & A_6 & 1 \end{bmatrix}$$

which is a function of A_2C_1. By induction, therefore, the permanent of such a matrix as (2) will depend only on

$$A_2C_1, A_3C_2, \cdots, A_nC_{n-1} .$$

Hence, without loss of generality, we may assume that the C's are all equal to 1 and by an obvious change in notation define the standard tridiagonal matrix by

$$M = M_n = M_n(a_1, a_2, \cdots, a_{n-1}) = \begin{bmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ a_1 & 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & a_2 & 1 & 1 & \cdots & 0 & 0 \\ 0 & 0 & a_3 & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & 0 & 0 & \cdots & a_{n-1} & 1 \end{bmatrix} .$$

We denote the permanent of this matrix M by

$$\Delta = \Delta_n = \Delta_n(a_1, a_2, \cdots, a_{n-1}) = \text{per}\, M_n(a_1, a_2, \cdots, a_{n-1}).$$

We also adopt the conventions

$$(3) \qquad \Delta_0 = 1 \quad \text{and} \quad \Delta_{-1} = 0 .$$

3. BASIC PROPERTIES

We begin with the basic recurrence for Δ_n.

Theorem 1. If $n \geqslant 1$,

$$\Delta_n(a_1, \cdots, a_{n-1}) = \Delta_{n-1}(a_1, \cdots, a_{n-2}) + a_{n-1}\Delta_{n-2}(a_1, \cdots, a_{n-3}).$$

Proof. This follows at once by expanding Δ_n by minors of the elements of the last column of $M_n(a_1, \cdots, a_{n-1})$. This recurrence is an efficient way of calculating successive Δ's when the a's are given. It is clear from (4) that Δ_n is linear in each of its independent variables $a_1, \cdots, a_{n-1}$. For future use we give Table 1 of Δ_n. We observe from this table that Δ_n is unaltered when its arguments are reversed. In general we have

Theorem 2. $\Delta_n(a_1, a_2, \cdots, a_{n-1}) = \Delta_n(a_{n-1}, a_{n-2}, \cdots, a_1).$

Proof. The theorem holds trivially for $n = 0, 1, 2.$ If true for $n-1$ amd $n-2$, (4) becomes

$$\Delta_n(a_1, a_2, \cdots, a_{n-1}) = \Delta_{n-1}(a_{n-2}, \cdots, a_1) + a_{n-1}\Delta_{n-2}(a_{n-3}, \cdots, a_1).$$

But the right-hand side is the result of expanding the permanent of $M_n(a_{n-1}, a_{n-2}, \cdots, a_1)$ by minors of elements of its first row. Hence the theorem is true for n and the induction is complete.

Table 1

n	$\Delta_n(a_1, a_2, \cdots, a_{n-1})$
-1	0
0	1
1	1
2	$1 + a_1$
3	$1 + a_1 + a_2$
4	$1 + a_1 + a_2 + a_3 + a_1a_3$
5	$1 + a_1 + a_2 + a_3 + a_4 + a_1a_3 + a_2a_4 + a_1a_4$
6	$1 + \sum_{i=1}^{5} a_i + a_1a_3 + a_2a_4 + a_3a_5 + a_1a_4 + a_2a_5 + a_1a_5 + a_1a_3a_5$

Since Δ_n is linear in each variable a_j one can ask what are the functions G_j and H_j in

(5) $$\Delta_n(a_1, \cdots, a_{n-1}) = G_j + H_ja_j \qquad (1 \leqslant j < n).$$

It is clear from (4) that when $j = n - 1$

$$G_{n-1} = \Delta_{n-1}(a_1, \cdots, a_{n-2}), \qquad H_{n-1} = \Delta_{n-2}(a_1, \cdots, a_{n-3}).$$

The general theorem is

Theorem 3. In (5),

$$G_j = \Delta_{n-j}(a_{j+1}, \cdots, a_{n-1})\Delta_j(a_1, \cdots, a_{j-1})$$
$$H_j = \Delta_{n-j-1}(a_{j+1}, \cdots, a_{n-1})\Delta_{j-1}(a_1, \cdots, a_{j-2}) .$$

Proof. This can be proved by expanding Δ_n by minors of the elements of its j^{th} column and using Laplacian development of these minors. However, a simpler proof is afforded by the introduction of the following generalized permanents $\Delta_{K,r}$ defined for $K \leqslant r$ by

(6) $$\Delta_{K,r} = \Delta_{K,r}(a_1, a_2, \cdots) = \Delta_K(a_{r-K+1}, a_{r-K+2}, \cdots, a_{r-1}) = \Delta_K(a_{r-1}, a_{r-2}, \cdots, a_{r-K+1}).$$

In particular we have

$$\Delta_{K,K} = \Delta_K(a_1, a_2, \cdots, a_{K-1}).$$

Theorem 1 applied to these two equivalent definitions gives us the following useful relations.

(7) $$\Delta_{K,r} = \Delta_{K-1,r} + a_{r-K+1}\Delta_{K-2,r}$$

(8) $$\Delta_{K,r} = \Delta_{K-1,r-1} + a_{r-1}\Delta_{K-2,r-2}.$$

We claim now that for $0 \leqslant K < n$

(9) $$\Delta_n = \Delta_{K,n}\Delta_{n-K} + a_{n-K}\Delta_{K-1,n}\Delta_{n-K-1} .$$

In fact this is trivial when $K = 0$ by (3) and (6) and when $K = 1$ it is a restatement of Theorem 1. To proceed inductively for K to $K + 1$ we note that

$$\Delta_{n-K} = \Delta_{n-(K+1)} + a_{n-(K+1)}\Delta_{n-1-(K+1)}$$

by Theorem 1. Substituting this into our induction hypothesis (9) we obtain

$$\Delta_n = \Delta_{n-(K+1)}\left\{\Delta_{K,n} + a_{n-K}\Delta_{K-1,n}\right\} + a_{n-(K+1)}\Delta_{K,n}\Delta_{n-1-(K+1)} .$$

But by (7) the quantity in the braces in $\Delta_{K+1,n}$. Hence our induction is complete. If now we put $K = n - j$ and $r = n$ in (6) and (9) the theorem follows.

As a corollary we have

$$\frac{\partial\Delta_n(a_1, a_2, \cdots, a_{n-1})}{\partial a_j} = \Delta_{j-1}(a_1, a_2, \cdots, a_{j-2})\Delta_{n-j-1}(a_{j+2}, \cdots, a_{n-1}).$$

4. CONNECTION WITH CONTINUED FRACTIONS

The ratio of two Δ's is the convergent of a continued fraction. More precisely we have

Theorem 4.

$$1+\frac{a_1|}{|1}+\frac{a_2|}{|1}+\cdots+\frac{a_{n-1}|}{|1} = \frac{\Delta_n(a_1, a_2, \cdots, a_{n-1})}{\Delta_{n-1}(a_2, a_3, \cdots, a_{n-1})} .$$

Proof. By Theorems 2 and 1 we may write

$$\frac{\Delta_n(a_1, \cdots, a_{n-1})}{\Delta_{n-1}(a_2, \cdots, a_{n-1})} = \frac{\Delta_n(a_{n-1}, \cdots, a_1)}{\Delta_{n-1}(a_{n-1}, \cdots, a_2)}$$

$$= \frac{\Delta_{n-1}(a_{n-1}, \cdots, a_2) + a_1\Delta_{n-2}(a_{n-1}, \cdots, a_3)}{\Delta_{n-1}(a_{n-1}, \cdots, a_2)} = 1+\frac{a_1}{\Delta_{n-1}/\Delta_{n-2}} .$$

Iterating this identity until we reach $\Delta_1/\Delta_0 = 1$, we obtain the theorem.

As an example, in case all the a's are equal to 1 we get the Fibonacci irrational

$$\theta = \tfrac{1}{2}(1+\sqrt{5}) = 1+\frac{1|}{|1}+\frac{1|}{|1}+\cdots$$

whose successive convergents

$$1, 2/1, 3/2, 5/3, 8/5, \cdots$$

are the ratios of consecutive Fibonacci numbers F_{n+1}/F_n. Hence

$$\Delta_n(1, 1, 1, \cdots, 1) = F_{n+1} \tag{10}$$

a fact which follows at once from (4). Conversely as soon as we have developed other formulas like (10) we can evaluate other continued fractions of Ramanujan type given in Theorem 4.

5. PERMANENTS WITH PERIODIC ELEMENTS

We are now prepared to consider the case in which the elements a of Δ are periodic of period p so that $a_{j+p} = a_j$. We shall find that the permanents

$$\Delta_s, \Delta_{s+p}, \Delta_{s+2p}, \cdots$$

constitute in this case a recurring series of the second order with constant coefficients depending only on p and the values of $a_1, a_2, \cdots, a_p$ but not depending on s. From this it will follow that Δ_n is a linear combination of two Lucas functions U_h and U_{h+1}, where $h = [n/p]$ whose coefficients now depend on $s = n - hp$. More precisely

$$U_h = U_h(P,Q) = (a^h - b^h)/(a-b),$$

where

$$P = a+b, \quad Q = ab$$

and

$$U_0 = 0, \quad U_1 = 1, \quad U_2 = P \tag{11}$$

and

$$U_h = PU_{h-1} - QU_h . \tag{12}$$

We denote the $n\times n$ permanent based on the periodic a's by

$$\Delta_n(\dot{a}_1, a_2, \cdots, \dot{a}_p)$$

so that (10) becomes

$$\Delta_n(\dot{1}) = F_{n+1} .$$

6. THE CASE $p = 1$

In this simple case we have

Theorem 5.

(13) $$\Delta_n(\dot{a}_1) = U_{n+1}(1, -a_1).$$

Proof. By (12),

$$U_{n+1}(1, -a_1) = U_n(1, -a_1) + a_1 U_{n-1}(1, -a_1).$$

But by (4),

$$\Delta_n(\dot{a}_1) = \Delta_{n-1}(\dot{a}_1) + a_1 \Delta_{n-2}(\dot{a}_1)$$

since $a_{n-1} = a_1$ for all n.

Hence both Δ_n and U_{n+1} satisfy the same recurrence. They also have the same starting values for $n = -1$ and $n = 0$. Hence the two functions coincide.

Corollary. $\Delta_{n-1}(\dot{a}) = \left\{(1+\sqrt{1+4a})^n - (1-\sqrt{1+4a})^n\right\}/(2^n\sqrt{1+4a})$.

Proof. Referring to (13) we see that a and b are roots of $x^2 - x - a = 0$. Examples of the Corollary are

$$\Delta_{n-1}(\dot{0}) = 1$$

$$\Delta_{n-1}(-\dot{1}) = \frac{\sqrt{12}}{3}\sin(\pi n/3)$$

$$\Delta_{n-1}(\dot{2}) = \left\{2^n - (-1)^n\right\}/3.$$

This last example leads, via Theorem 4, to

$$1 + \frac{2|}{|1} + \frac{2|}{|1} + \frac{2|}{|1} + \cdots = 2$$

as is easily verified.

7. THE CASE $p = 2$

This case is also relatively simple. We have

Theorem 6. $\Delta_n(\dot{a}_1, \dot{a}_2) = (1 + a_1 + a_2)\Delta_{n-2}(\dot{a}_1, \dot{a}_2) - a_1 a_2 \Delta_{n-4}(\dot{a}_1, \dot{a}_2)$.

Proof. First suppose n is odd so that $a_{n-1} = a_2$. Then Theorem 1 gives

$$\Delta_n = \Delta_{n-1} + a_2\Delta_{n-2} = \Delta_{n-2} + a_1\Delta_{n-3} + a_2\Delta_{n-2}.$$

But $$\Delta_{n-3} = \Delta_{n-2} - a_2\Delta_{n-4}.$$

Elimination of Δ_{n-3} gives the theorem for n odd. If n is even, we simply interchange the roles of a_1 and a_2.

The counterpart of Theorem 5 for $p = 2$ is

Theorem 7. $$\Delta_{2n}(\dot{a}_1, \dot{a}_2) = U_{n+1}(1 + a_1 + a_2, a_1a_2) - a_2 U_n(1 + a_1 + a_2, a_1 a_2)$$

$$\Delta_{2n+1}(\dot{a}_1\dot{a}_2) = U_{n+1}(1 + a_1 + a_2, a_1a_2).$$

Proof. Let $W_n = \Delta_{2n}(\dot{a}_1, \dot{a}_2)$. By Theorem 6

$$W_n = (1 + a_1 + a_2)W_{n-1} - a_1a_2W_{n-2}$$

with

$$W_0 = 1, \quad W_1 = \Delta_2(a_1, a_2) = 1 + a_1.$$

But

$$U_{n+1}(1 + a_1 + a_2, a_1a_2) - a_2U_n(1 + a_1 + a_2, a_1a_2)$$

enjoys the same recurrence and the same initial conditions. This proves the first part of the Theorem. The second part is proved in the same way.

We note that, unlike $\Delta_{2n}(a_1, a_2)$, the function $\Delta_{2n+1}(\dot{a}_1, \dot{a}_2)$ is symmetric in a_1 and a_2.

Examples of Theorem 7 are

$$\Delta_{2n+1}(\dot{0}, \dot{1}) = 2^n, \quad \Delta_{2n}(\dot{0}, \dot{1}) = 2^{n-1}, \quad \Delta_{2n}(\dot{1}, \dot{0}) = 2^n$$

$$\Delta_{2n+1}(\dot{1}, -\dot{1}) = F_{n+1}, \quad \Delta_{2n}(\dot{1}, -\dot{1}) = F_{n+2}, \quad \Delta_{2n}(-\dot{1}, \dot{1}) = F_{n-1}$$

$$\Delta_{2n+1}(\dot{\omega}, \dot{\omega}^2) = \tfrac{1}{2} i^n (1+(-1)^n), \quad \Delta_{4r}(\dot{\omega}, \dot{\omega}^2) = (-1)^n$$

$$\Delta_{2n+1}(-\dot{\omega}, \dot{\omega}^2) = n+1, \quad \Delta_{2n}(-\dot{\omega}, -\dot{\omega}^2) = 1 - n\omega$$

$$\Delta_{2n-1}(i, -i) = \frac{\sqrt{12}}{3} \sin(\pi n/3)$$

$$\Delta_{2n-1}(\dot{1}, \dot{2}) = \frac{(2+\sqrt{2})^n - (2-\sqrt{2})^n}{2\sqrt{2}}, \quad \Delta_{2n}(\dot{1}, \dot{2}) = \frac{(2+\sqrt{2})^n + (2-\sqrt{2})^n}{2}$$

Here

$$\omega = e^{2\pi i/3} = \frac{-1+\sqrt{3}\,i}{2} \quad .$$

The last two results easily lead to

$$1 + \frac{1|}{|1} + \frac{2|}{|1} + \frac{1|}{|1} + \frac{2|}{|1} + \cdots = \sqrt{2} \;.$$

Inspection of the above examples shows them to behave exponentially, linearly or periodically as $n \to \infty$. This is a general fact, true of periodic a's of any period length p.

8. THE GENERAL PERIODIC CASE

We now take up the complicated general case of $p \geqslant 3$, although the theorems we are about to obtain hold for $p = 1$ and 2. For this purpose we enlarge the definition (6) of $\Delta_{K,r}$ to include the cases $K > r$. That is, we define for the periodic case

$$\Delta_{K,r}(\dot{a}_1, a_2, \cdots, \dot{a}_p) = \Delta_K(a_{r-K+1}, a_{r-K+2}, \cdots, a_{r-1}),$$

where the subscripts of the a's are to be interpreted modulo p. Thus if $p = 4$,

$$\Delta_{5,2}(\dot{a}_1, a_2, a_3, \dot{a}_4) = \Delta_5(a_{-2}, a_{-1}, a_0, a_1) = \Delta_5(a_2, a_3, a_4, a_1)$$

$$\Delta_{4,1}(\dot{a}_1, a_2, a_3, \dot{a}_4) = \Delta_4(a_{-2}, a_{-1}, a_0) = \Delta_4(a_2, a_3, a_4)$$

$$\Delta_{3,0}(\dot{a}_1, a_2, a_3, \dot{a}_4) = \Delta_3(a_2, a_3) \;.$$

It is easily verified that

$$\Delta_{5,2}(\dot{a}_1, a_2, a_3, \dot{a}_4) = \Delta_{4,1} + a_1 \Delta_{3,0}$$

which for $K = 5$ and $r = 2$ is a particular case of (7). Formulas (7) and (8) are still true in general by Theorem 1.

Theorem 8. For $0 \leqslant s < p$ let

$$A(p,s) = \Delta_{p,s} + a_s \Delta_{p-2,s-1}$$

$$B(p,s) = a_s(\Delta_{p,s}\Delta_{p-2s-1} - \Delta_{p-1,s}\Delta_{p-1,s-1}).$$

Then if $n \equiv s \pmod p$,

$$\Delta_{n+p} = A(p,s)\Delta_n - B(p,s)\Delta_{n-p} \;,$$

where the argument in all the Δ's is $(\dot{a}_1, a_2, \cdots, \dot{a}_p)$.

Proof. Let $n = ph + s$. If in (9) we set $K = p$ and use the fact that $a_{n+i} = a_{s+i}$ we get

$$\Delta_{ph+s} = \Delta_{p,s}\Delta_{p(h-1)+s} + a_s \Delta_{p-1,s}\Delta_{p(h-1)+s-1} \;. \tag{14}$$

In the same way replacing n by $n - p$ and setting $K = p - 1$ we have

$$\Delta_{p(h-1)+s-1} = \Delta_{p-1,s-1}\Delta_{p(h-2)+s} + a_s \Delta_{p-2,s-1}\Delta_{p(h-2)+s-1} \;. \tag{15}$$

Beginning with (14) and continually applying (15) gives the following for Δ_n

$$\Delta_{ph+s} = \Delta_{p,s}\Delta_{p(h-1)+s} + \Delta_{p-1,s}\Delta_{p-1,s-1} \sum_{\mu=1}^{h-1} a_s^{\mu} \left\{ \Delta_{p-2,s-1} \right\}^{\mu-1} \Delta_{p(h-\mu-1)+s} + \Delta_{p-1,s} a_s^h \left\{ \Delta_{p-2,s-1} \right\}^{h-1} \Delta_{s-1} \,.$$

(16)
$$\Delta_{p-1,s}\Delta_{p-1,s-1}\sum_{\mu=1}^{h-1} a^{\mu}\left\{\Delta_{p-2,s-1}\right\}^{\mu-1}\Delta_{p(h-\mu-1)+s}$$
$$= \Delta_{ph+s} - \Delta_{ps}\Delta_{p(h-1)+s} - a_s^h\Delta_{p-1,s}\Delta_{s-1}\left\{\Delta_{p-2,s-1}\right\}^{h-1}.$$

Next we multiply both sides of (16) by $a_s\Delta_{p-2,s-1}$ and add

$$a_s\Delta_{p-1,s}\Delta_{p-1,s-1}\Delta_{p(h-1)+s}$$

to both sides. If we subtract this result from (16) when h is replaced by $h+1$ we get

$$\Delta_{p(h+1)+s} - \Delta_{p,s}\Delta_{ph+s} = a_s\left\{\Delta_{p-2,s-1}\Delta_{ph+s} - \Delta_{p,s}\Delta_{p-2,s-1}\Delta_{p(h-1)+s} + \Delta_{p-1,s-1}\Delta_{p-1,s}\Delta_{p(h-1)+s}\right\}.$$

Collecting the coefficients of Δ_{ph+s} and $\Delta_{p(h-1)+s}$ gives us the theorem.

Our next goal is to show that $A(p,s)$ and $B(p,s)$ depend on p but not on s.

Theorem 9. $$B(p,s) = (-1)^p a_1a_2\cdots a_p.$$

Proof. It will suffice to show that

(17) $$\Delta_{p,s}\Delta_{p-2,s-1} - \Delta_{p-1}\Delta_{p-1,s-1} = (-1)^p a_{s-1}a_{s-2}\cdots a_{s-p+1},$$

where the subscripts on the a's are to be taken modulo p, because then, by definition of $B(p,s)$ we

$$B(p,s) = (-1)^p a_sa_{s-1}\cdots a_{s-p+1} = (-1)^p a_1a_2\cdots a_p.$$

To prove (17) we note that it holds for $p = 1$ since the left member is -1 and the product of a's is vacuous. Assuming the result holds for p and noting that (7) gives

$$\Delta_{p+1,3} = \Delta_{p,s} + a_{s-p}\Delta_{p-1,s}$$

and

$$\Delta_{p,s-1} - \Delta_{p-1,s-1} = a_{s-p}\Delta_{p,s}\Delta_{p-2,s-1}.$$

We have

$$\begin{aligned}\Delta_{p+1,s}\Delta_{p-1,s-1} - \Delta_{p,s}\Delta_{p,s-1} &= -\Delta_{p,s}[\Delta_{p,s-1} - \Delta_{p-1,s-1}] \\ &\quad + a_{s-p}\Delta_{p-1,s}\Delta_{p-1,s-1} \\ &= -a_{s-p}[\Delta_{p,s}\Delta_{p-2,s-1} - \Delta_{p-1,s}\Delta_{p-1,s-1}] \\ &= (-1)^{p+1}a_{s-1}a_{s-2}\cdots a_{s-p+1}a_{s-p}.\end{aligned}$$

Hence (17) holds for $p+1$ and the induction is complete.

Theorem 10. $A(p,s)$ is not a function of s.

Proof. Using both (7) and (8) with $k = p$ and $r = s$ and $s = 1$ we have

$$\begin{aligned}A(p,s) &= \Delta_{p,s} + a_s\Delta_{p-2,s-1} = a_s\Delta_{p-2,s-1} + \Delta_{p-1,s-1} + a_{s-1}\Delta_{p-2,s-2} \\ &= a_{s-1}\Delta_{p-2,s-2} + \Delta_{p,s-1} = A(p,s-1).\end{aligned}$$

Hence $A(p,s)$ does not depend on s.

We can write

(18) $$A(p,s) = A(p,p) = P_p = P = \Delta_p(a_1,\cdots,a_{p-1}) + a_p\Delta_{p-2}(a_2,\cdots,a_{p-2})$$

and

(19) $$Q_p = Q = (-1)^p a_1a_2\cdots a_p$$

and restate Theorem as follows

Theorem 11. $$\Delta_{n+p} = P\Delta_n - Q\Delta_{n-p}.$$

Armed with this information we can at once evaluate $\Delta_n(a_1,\cdots,a_p)$ as a linear combination of two consecutive members of the Lucas sequence $\left\{U_m(P,Q)\right\}$ as follows.

Theorem 12.

(20) $$\Delta_{hp+s} = \Delta_sU_{h+1}(P,Q) + (\Delta_{p+s} - P\Delta_s)U_h(P,Q).$$

Proof. This relation holds for $h = 0$ and, since $U_2(P,Q) = P$ for $h = 1$. By Theorems 11 and 12 both sides enjoy the same recurrence. Hence they coincide.

9. MORE ON THE FUNCTION P

The function

$$P = P_p(a_1, a_2, \cdots, a_p)$$

defined by (18) is not as simple as Q. We already know that

$$P_1 = 1 \quad \text{and} \quad P_2 = 1 + a_1 + a_2 .$$

We can tabulate P_p as follows

Table 2

p	$P_p(a_1, a_2, \cdots, a_p)$
1	1
2	$1 + a_1 + a_2$
3	$1 + a_1 + a_2 + a_3$
4	$1 + a_1 + a_2 + a_3 + a_4 + a_1a_3 + a_2a_4$
5	$1 + \sum_{i=1}^{5} a_i + \sum_{i=1}^{3} a_ia_{i+2} + \sum_{i=1}^{2} a_ia_{i+3}$
6	$1 + \sum_{i=1}^{6} a_i + \sum_{i<j\leqslant 6} a_ia_j - \sum_{i=1}^{5} a_ia_{i+1} + \sum_{i=1}^{2} a_ia_{i+2}a_{i+4}$

Further entries in this table are left to the curiosity of the reader. It will be observed that the entries cease to be symmetric functions of the a's with $p = 4$.

10. FIBONACCI-TYPE Δ'S

The permanent of a tridiagonal matrix with periodic a's will depend on Fibonacci numbers if we can make $P = 1$ and $Q = -1$ since

$$U_m(1, -1) = F_m .$$

For $p = 3$ this requires

$$P_3 = 1 + a_1 + a_2 + a_3 = 1, \qquad -Q_3 = a_1a_2a_3 = 1.$$

This means that the three a's are the roots any cubic equation of the form

$$x^3 + cx - 1 = 0. \tag{21}$$

The simplest example is $c = 0$ for which

$$a_1 = 1, \quad a_2 = \omega, \quad a_3 = \omega^2$$

or some other permutation of these. For this case Theorem 12 gives the examples

$$\Delta_{3h}(\dot{1}, \omega, \dot{\omega}^2) = F_{h+1} - \omega^2 F_n$$
$$\Delta_{3h+1}(\dot{1}, \omega, \dot{\omega}^2) = F_{h+1} + \omega^2 F_n$$
$$\Delta_{3h+2}(\dot{1}, \omega, \dot{\omega}^2) = 2F_{h+1} .$$

Another special case is that of $c = -2$ in which the roots of (21) are -1 and the two Fibonacci irrationals, for example

$$a_1 = \theta, \qquad a_2 = \bar{\theta}, \qquad a_3 = -1.$$

For this choice we get

$$\Delta_{3h}(\dot{\theta}, \bar{\theta}, -\dot{1}) = F_{h+2}$$
$$\Delta_{3h+1}(\dot{\theta}, \bar{\theta}, -\dot{1}) = F_{h+1} - \theta F_h$$
$$\Delta_{3h+2}(\dot{\theta}, \bar{\theta}, -\dot{1}) = (1+\theta)F_{h+1}.$$

The reader may wish to write such formulas for other permutations of $\theta, \bar{\theta}, -1$.

For $p = 4$ our requirement becomes

(22) $$a_1 + a_2 + a_3 + a_4 + a_1a_3 + a_2a_4 = 0, \qquad a_1a_2a_3a_4 = -1.$$

Examples are

$$a_1 = i, \quad a_2 = -1, \quad a_3 = -i, \quad a_4 = 1, \quad a_1 = \omega, \quad a_2 = \theta, \quad a_3 = \omega^2, \quad a_4 = \bar{\theta}.$$

More general examples are

$$a_1 = \tfrac{1}{2}(-t + \sqrt{t^2 - 4\epsilon}), \qquad a_2 = \tfrac{1}{2}(t + \sqrt{t^2 + 4\epsilon}),$$
$$a_3 = \tfrac{1}{2}(-t - \sqrt{t^2 - 4\epsilon}), \qquad a_4 = \tfrac{1}{2}(t - \sqrt{t^2 + 4\epsilon}),$$

where t is any real or complex parameter and $\epsilon^2 = 1$. In any case there are eight permutations of the four a's that maintain (22). These are, in cycle notation

$$(1)(2)(3)(4), \quad (1)(3)(24), \quad (13)(2)(4), \quad (13)(24)(12)(34), \quad (14)(23), \quad (1234), \quad (1432).$$

With any one of these choices we have for $\Delta_n = \Delta_n(\dot{a}_1, a_2, a_3, \dot{a}_4)$

$$\Delta_{4h} = F_{h+1} - a_4(1+a_2)F_h$$
$$\Delta_{4h+1} = F_{h+1} - a_1a_4F_h$$
$$\Delta_{4h+2} = (1+a_1)F_{h+1} - a_1a_2a_4F_h$$
$$\Delta_{4h+3} = (1+a_1+a_2)F_{h+1}.$$

Instead of forcing Δ_n to involve the Fibonacci numbers we can make it a linear function of n by choosing $P = 2$ and $Q = 1$ because $U_n(2,1) = n$.

For $p = 3$ the conditions become

(23) $$a_1 + a_2 + a_3 = 1, \qquad a_1a_2a_3 = -1.$$

One obvious solution is to choose two of the a's equal to 1 and the third -1. Thus we find

$$\Delta_{3h}(\dot{1}, 1, -\dot{1}) = 2h+1, \quad \Delta_{3h+1}(\dot{1}, 1, -\dot{1}) = 1, \quad \Delta_{3h+2}(\dot{1}, 1, -\dot{1}) = 2h+2, \quad \Delta_{3h}(\dot{1}, -1, \dot{1}) = 1,$$
$$\Delta_{3h+1}(\dot{1}, -1, \dot{1}) = 2h+1, \quad \Delta_{3h+2}(\dot{1}, -1, \dot{1}) = 2h+2, \quad \Delta_{3h}(-\dot{1}, 1, \dot{1}) = 1, \quad \Delta_{3h+1}(-\dot{1}, 1, \dot{1}) = 1, \quad \Delta_{3h+2}(-\dot{1}, 1, \dot{1}) = 0.$$

Another choice of a's satisfying (23) is any permutation of

$$-2\cos(2\pi/7), \quad -2\cos(4\pi/7), \quad -2\cos(6\pi/7).$$

The most general solutions of (23) are of course the roots of

$$x^3 - x^2 + cx + 1 = 0$$

and this leads to the linear function

$$\Delta_{3h+s} = \Delta_s + (\Delta_{s+3} - \Delta_s)h.$$

The reader may have observed in the above that, of all the formulas for Δ_{hp+s}, the simplest is that for $s = p - 1$. The reason for this phenomenon is to be seen by substituting $s = -1$ in Theorem 12. We obtain simply

$$\Delta_{hp-1} = \Delta_{p-1}U_p(P,Q).$$

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