A GENERALIZED PASCAL'S TRIANGLE

C. K. WONG*†and T. W. MADDOCKS**
University of Illinois, Urbana, Illinois 61801

1. INTRODUCTION

In the study of a combinatorial minimization problem related to multimodule computer memory organizations [5], a triangle of numbers is constructed, which enjoys many of the pleasant properties of Pascal's triangle [1,2]. These numbers originate from counting a set of points in the k-dimensional Euclidean space.

In this paper we only list some of the properties which are similar to those associated with Pascal's triangle. Other properties will be the subject of further investigation.

2. r-SPHERES IN RECTILINEAR METRIC

Let

$$U_r^{(k)} = \left\{ (x_1, x_2, \cdots, x_k) \middle| x_i \text{ integers, } i = 1, 2, \cdots, k, \text{ and } \sum_{i=1}^{k} |x_i| \leqslant r \right\}.$$

The aim of this section is to obtain a formula for the cardinality $|U_r^{(k)}|$ of the set $U_r^{(k)}$.

Lemma 1. Let

$$S_j^{(k)} = \left\{ (x_1, x_2, \cdots, x_k) \middle| x_i \text{ integers, } i = 1, 2, \cdots, k \text{ and } \sum_{i=1}^{k} |x_i| = j \right\},$$

then

$$|S_j^{(k)}| = \begin{cases} 1, & j = 0, \\ \displaystyle\sum_{i=0}^{k-1} \binom{k}{i} \binom{j-1}{k-i-1} 2^{k-i}, & j \geqslant 1. \end{cases}$$

Proof. Note that the number of ways to place j nondistinct objects into k distinct cells is $\binom{k+j-1}{k-1}$. (See [3].) Consequently, the number of ways to place j nondistinct objects into k distinct cells such that none of them is empty is $\binom{j-1}{k-1}$. In $S_j^{(k)}$, if we group together all points $(x_1, x_2, \cdots, x_k)$ which have the same number of zero coordinates, the result follows.

Theorem 1.

$$|U_r^{(k)}| = \sum_{i=0}^{k} \binom{k}{i} \binom{r}{k-i} 2^{k-i}.$$

Proof. It follows from

$$|U_r^{(k)}| = 1 + \sum_{j=1}^{r} |S_j^{(k)}| \quad \text{and} \quad \sum_{i=0}^{n} \binom{i}{a} = \binom{n+1}{a+1}.$$

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**NSF Academic Year Institute University of Illinois Fellow in Mathematics, 1972-73.
†Current address: I.B.M. T.J. Watson Research Center, Yorktown Heights, N.Y. 10598.

The numbers $|S_j^{(k)}|$ and $|U_r^{(k)}|$ have the following geometric interpretation:

It suffices to mention the case $k = 2$. Partition the Euclidean plane into unit squares. Fix any square as the origin, which will be called the 0^{th} sphere. All squares which have at least one edge in common with the origin form the 1^{st} sphere. All those with at least one edge in common with a square in the 1^{st} sphere form the 2^{nd} sphere, and so on.

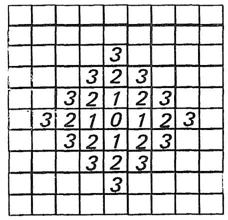

Figure 1

The numbers in Fig. 1 indicate what spheres the squares are in. $|S_j^{(2)}|$ is then the number of squares comprising the j^{th} sphere, i.e. its "surface area," and $|U_r^{(2)}|$ is the number of squares constituting the r^{th} sphere and its interior, i.e., its "volume."

The generalization to $k > 2$ is clear.

3. A GENERALIZED PASCAL'S TRIANGLE

For simplicity, let us write $M_{k,r}$ for $|U_r^{(k)}|$. We then have the following observations:

Theorem 2. (i)
$$M_{k,r} = M_{r,k}$$

(ii)
$$M_{k+1,r} = 2\sum_{j=0}^{r-1} M_{k,j} + M_{k,r}$$

(iii)
$$M_{k+1,r+1} = M_{k+1,r} + M_{k,r+1} + M_{k,r}$$

Proof. (i) If $k \geqslant r$, then
$$M_{k,r} = \sum_{i=k-r}^{k} \binom{k}{i}\binom{r}{k-i} 2^{k-i} = \sum_{j=0}^{r} \binom{k}{j+k-r}\binom{r}{r-j} 2^{r-j} = \sum_{j=0}^{r} \binom{r}{j}\binom{k}{r-j} 2^{r-j} = M_{r,k}.$$

Similarly for the case $k < r$.

(ii)
$$2\sum_{j=0}^{r-1} M_{k,j} = \sum_{i=0}^{k} \binom{k}{i}\binom{r}{k+1-i} 2^{k+1-i}$$

$$2\sum_{j=0}^{r-1} M_{k,j} + M_{k,r} = \binom{k}{0}\binom{r}{k+1} 2^{k+1} + \sum_{i=1}^{k}\binom{k}{i}\binom{r}{k+1-i} 2^{k+1-i} + \sum_{j=0}^{k-1}\binom{k}{j}\binom{r}{k-j} 2^{k-j} + \binom{k}{k}\binom{r}{0} 2^0$$

$$= \binom{k+1}{0}\binom{r}{k+1} 2^{k+1} + \sum_{i=1}^{k}\left[\binom{k}{i} + \binom{k}{i-1}\right]\binom{r}{k+1-i} 2^{k+1-i} + \binom{k+1}{k+1}\binom{r}{0} 2^0$$

$$= \sum_{i=0}^{k+1}\binom{k+1}{i}\binom{r}{k+1-i} 2^{k+1-i} = M_{k+1,r}.$$

(iii) It follows directly from (ii).

Theorem 3. For $k = 0, 1, 2, \cdots$, let

$$S_k = \begin{cases} \displaystyle\sum_{i=0}^{(k-1)/2} M_{k-2i,i}\,, & \text{if } k \text{ is odd,} \\ \displaystyle\sum_{i=0}^{k/2} M_{k-2i,i}\,, & \text{if } k \text{ is even.} \end{cases}$$

Then

$$S_{k+3} = S_k + S_{k+1} + S_{k+2}$$

for $k = 0, 1, 2, \cdots$.

In other words, they form a Tribonacci sequence.

Proof. It follows from (iii) of Theorem 2.

We can now construct a triangle of the numbers $\{M_{k,r}\}$, $k, r = 0, 1, 2, \cdots$. The n^{th} row consists of the numbers $\{M_{n-i,i}\}$ in the order of $i = 0, 1, 2, \cdots, n$ from left to right. The left diagonals thus consist of numbers with fixed r, and the right diagonals numbers with fixed k.

By (iii) of Theorem 2, each number in the n^{th} row is the sum of the three adjacent numbers in the $(n-1)^{st}$ and $(n-2)^{nd}$ rows. For example, the number 25 in the 5^{th} row is the sum of its three adjacent numbers 5, 7, 13 in the 3^{rd} and 4^{th} rows. Therefore, instead of using the formula in Theorem 1, we can fill in a row by adding appropriate numbers in the two preceding rows. Finally, by Theorem 3, the sums of the more gently sloping diagonals form the Tribonacci sequence, 1, 1, 2, 4, 7, 13, 24, 44, 81, 149, ⋯.

The first 10 rows of this generalized Pascal's triangle is displayed below.

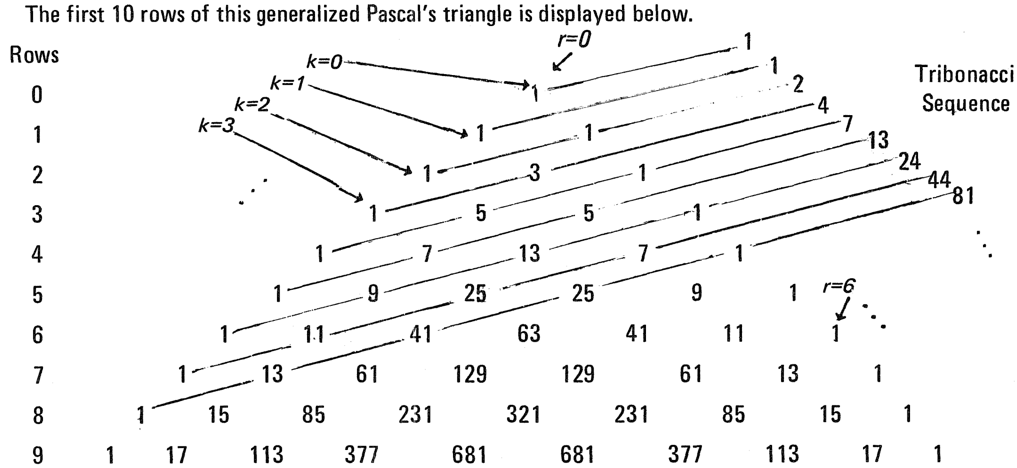

Figure 2

ACKNOWLEDGEMENT

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