A GREATEST INTEGER THEOREM FOR FIBONACCI SPACES

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1. INTRODUCTION

If $S = \{ s_j \}$ is any integer sequence of a Fibonacci space [2] based on a polynomial

$$f(x) = -a_0 - \cdots - a_{n-1}x^{n-1} + x^n = (x - r_1) \cdots (x - r_n),$$

$a_i \in Z,\ r_1$ real, r_i distinct, $|r_i| < 1$ for $i \geqslant 2$, then

$$[r_1^k s_\ell + F] = s_{k+\ell}$$

with any fixed k, and F on (0,1), for all ℓ sufficiently large. This is a broad generalization, in an asymptotic sense, of a conjecture by D. Zeitlin [3] concerning the case

$$f(x) = -1 - Mx + x^2, \quad M \geqslant 1, \quad F = M/(M+1), \quad \text{and} \quad S = \{0,1,M,\cdots\},$$

defined by $u_\ell + Mu_{\ell+1} = u_{\ell+2}$. The latter is shown to be true in all cases but one, and in slightly revised form in the remaining case.

2. A GENERAL ASYMPTOTIC THEOREM

With the polynomial

$$f(x) = -a_0 - a_1x - \cdots - a_{n-1}x^{n-1} + x^n = (x - r_1) \cdots (x - r_n),$$

a_i integers, r_1 real, r_i distinct, $|r_i| < 1$ for $i \geqslant 2$, we associate the n-space $C(f)$ of all (complex) sequences $S = \{s_0, s_1, \cdots\}$ in which $s_0, \cdots, s_{n-1}$ are arbitrary, but having

$$a_0s_j + \cdots + a_{n-1}s_{j+n-1} = s_{j+n}; \quad j \geqslant 0.$$

The n geometric sequences

$$R_i = \{1, r_i, r_i^2, \cdots\}$$

form a basis for the space $C(f)$, in terms of which an arbitrary integral sequence S may be expressed in the form

$$S = c_1R_1 + \cdots + c_nR_n, \qquad \text{i.e.,} \qquad s_\ell = c_1r_1^\ell + \cdots + c_nr_n^\ell; \quad \ell \geqslant 0.$$

Since $|r_i| < 1,\ i \geqslant 2$, we may write

(1) $$s_\ell = c_1r_1^\ell + e_\ell; \qquad e_\ell \to 0.$$

These results may be found in [2]. That c_1 (and hence e_ℓ) are real is shown in an Appendix. As an immediate consequence, we have the asymptotic

Theorem 1. Let F be an arbitrary constant on the open interval (0,1), and $S = \{ s_j \}$ an integral sequence of the space $C(f)$. Then for fixed $k \geqslant 0$, one has the greatest integer

$$[r_1^k s_\ell + F] = s_{k+\ell}$$

for all ℓ sufficiently large.

Proof. Using (1), we have only to prove

$$c_1r_1^{k+1} + e_{k+\ell} \leqslant r_1^k(c_1r_1^\ell + e_\ell) + F < c_1r_1^{k+\ell} + e_{k+\ell} + 1$$

for large ℓ, i.e.,

$$e_{k+\ell} - r_1^k e_\ell \leqslant F < e_{k+\ell} - r_1^k e_\ell + 1$$

and this is obvious since $e_\ell \to 0$ and $0 < F < 1$.

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3. THE ZEITLIN CONJECTURE

For the integer $M \geqslant 1$, let

$$f(x) = 1 - Mx + x^2 = (x-a)(x-b), \quad a > b, \quad \text{and} \quad F = M/(M+1).$$

The roots a,b have the properties

$$a > M, \quad b < 0, \quad |b| = (\rho - M)/2 < 1, \quad ab = -1, \quad a - b = \rho; \quad \rho \equiv (M^2+4)^{1/2}.$$

The sequence $U = \left\{ u_0, u_1, \cdots \right\}$ is defined recursively by

$$u_0 = 0, \quad u_1 = 1, \quad u_\ell + Mu_{\ell+1} = u_{\ell+2}; \quad \ell \geqslant 0,$$

and is well known [2], p. 103, to be related to the roots by

$$u_\ell = \rho^{-1}(a^\ell - b^\ell); \quad \ell \geqslant 0.$$

From this we find

$$a^k u_\ell = \rho^{-1}(a^{k+\ell} - b^{k+\ell}) - \rho^{-1}b^\ell(a^k - b^k),$$

or

(2) $$a^k u_\ell = u_{k+\ell} - b^\ell u_k .$$

Theorem 2. For the sequence U, one has the greatest integer

$$[a^k u_\ell + F] = u_{k+\ell}$$

for $\ell \geqslant 2$, $k = 1$, and for $\ell \geqslant k \geqslant 2$ except possibly in the case ℓ odd $\geqslant k$ odd $\geqslant 3$ when $M \geqslant 2$.

Proof. We only sketch the argument, which closely follows that in [1]. In all cases, the final verification consists in the laborious comparison of two polynomials in M, for $M \geqslant 1$. The required relation

$$u_{k+\ell} \leqslant a^k u_\ell + F < u_{k+\ell} + 1$$

is seen from (2) to be equivalent to

$$-1/(M+1) < b^\ell u_k \leqslant M/(M+1).$$

Case I. $\ell \geqslant 2$, $k = 1$. For ℓ even, it suffices to prove $b^2 \leqslant M/(M+1)$. For ℓ odd, $|b|^3 < 1/(M+1)$ suffices. These are found to hold upon replacing $|b|$ by its value $(\rho - M)/2$ and rationalizing.

Case II. $\ell \geqslant k \geqslant 2$. For ℓ, k even, it suffices to show $b^k u_k \leqslant M/(M+1)$. But

$$b^k u_k = b^k \rho^{-1}(a^k - b^k) = \rho^{-1}(1 - b^{2k}) < M/(M+1)$$

will hold for all k iff $\rho^{-1} < M/(M+1)$, which is verified as before.

For ℓ even $\geqslant k$ odd $\geqslant 2$, $b^{k+1}u_k \leqslant M/(M+1)$ suffices. Now,

$$b^{k+1}u_k \equiv |b|\rho^{-1}(1 + b^{2k})$$

by an ananogous step, so we need only show that

$$|b|\rho^{-1}(1 + b^6) \leqslant M/(M+1).$$

This is the most laborious verification.

For ℓ odd $\geqslant k$ even $\geqslant 2$, it suffices to prove $-b^{k+1}u_k < 1/(M+1)$. Here we find

$$-b^{k+1}\rho^{-1}(a^k - b^k) = |b|\rho^{-1}(1 - b^{2k}) < 1/(M+1).$$

since in the limit, $|b|\rho^{-1} < 1/(M+1)$. This is easy.

Finally, suppose ℓ odd $\geqslant k$ odd $\geqslant 2$, and $M = 1$. It suffices to prove

$$-b^k u_k \equiv \rho^{-1}(1 + b^{2k}) < 1/(M+1), \quad k \geqslant 3,$$

and this is true since $\rho^{-1}(1 + b^6) < 1/(M+1)$ is verifiable when $M = 1$ (and only then).

The relation of Theorem 2 may fail in the remaining case, as is easily seen from the example $M = 2$, $\ell = k = 3$, where

$$[a^3 u_3 + F] = 71 = 1 + u_6 .$$

Indeed it always fails for $M \geqslant 2$, $\ell = k$ odd $\geqslant 3$, as appears in the final

Theorem 3. For the sequence U, with $M \geqslant 2$, ℓ odd $\geqslant k$ odd $\geqslant 2$, the value of $[a^k u_\ell + F]$ is either $u_{k+\ell}$ or $u_{k+\ell} + 1$, according as $|b|^\ell u_k < 1/(M+1)$ or $1/(M+1) \leqslant |b|^\ell u_k$, the latter always obtaining for $\ell = k$.

Proof. Using (2), the relations of the theorem are found to be equivalent, respectively, to

$$-M/(M+1) \leqslant |b|^{\ell} u_k < 1/(M+1) \quad \text{and} \quad 1/(M+1) \leqslant |b|^{\ell} u_k < (M+2)/(M+1).$$

We note first that $|b|^{\ell} u_k$ is always between $-M/(M+1)$ and $(M+2)/(M+1)$. The first is obvious. For the second, it suffices to prove $|b|^k u_k < (M+2)/(M+1)$, k odd $\geqslant 3$. But

$$|b|^k u_k = \rho^{-1}(1+b^{2k}) \leqslant (M+2)/(M+1)$$

holds provided

$$\rho^{-1}(1+b^6) < (M+2)/(M+1),$$

which may be verified as in Theorem 2, Case II, second part.

Hence for fixed k, we consider the relation of $|b|^{\ell} u_k$ to $1/(M+1)$ as ℓ increases from k. Now if at the start we had

$$|b|^k u_k \equiv \rho^{-1}(1+b^{2k}) < 1/(M+1),$$

this would imply $\rho^{-1} < 1/(M+1)$, which is false for all $M \geqslant 2$. The theorem follows.

APPENDIX

Reality of c_1, e_{ℓ}

From [2] we write

(3)
$$\begin{vmatrix} R_1 \\ \vdots \\ R_n \end{vmatrix} = \begin{vmatrix} 1 & r_1 & \cdots & r_1^{n-1} \\ \vdots & & & \\ 1 & r_n & \cdots & r_n^{n-1} \end{vmatrix} \begin{vmatrix} U_0 \\ \vdots \\ U_{n-1} \end{vmatrix},$$

where

$$U_0 = \{1, 0, \cdots, 0, a_0, \cdots\}, \cdots, U_{n-1} = \{0, 0, \cdots, 1, a_{n-1}, \cdots\}$$

is an obvious basis, and the matrix determinant Δ is that of Vandermonde. Inversion gives

(4)
$$\begin{vmatrix} U_0 \\ \vdots \\ U_{n-1} \end{vmatrix} = \begin{vmatrix} r_{01} & \cdots & r_{0n} \\ \vdots & & \vdots \\ r_{n-1,1} & \cdots & r_{n-1,n} \end{vmatrix} \begin{vmatrix} R_1 \\ \vdots \\ R_n \end{vmatrix},$$

where

$$r_{jk} = (-1)^{j+k} R_{kj}/\Delta,$$

and R_{kj} is the k,j-minor of the matrix in (3). Since

$$S = |s_0 \cdots s_{n-1}| \cdot \begin{vmatrix} U_0 \\ \vdots \\ U_{n-1} \end{vmatrix} = |s_0 \cdots s_{n-1}| \cdot |r_{jk}| \cdot \begin{vmatrix} R_1 \\ \vdots \\ R_n \end{vmatrix}$$

we see that

$$c_1 = s_0 r_{01} + \cdots + s_{n-1} r_{n-1,1},$$

involving the first column of the inverse in (4). But each $r_{j,1}$ involves the quotient R_{1j}/Δ. The latter is real, since any complex roots r_i occur in pairs of conjugates.

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