ANOTHER PROPERTY OF MAGIC SQUARES

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1. INTRODUCTION

Consider $n \times n$ matrices $A = [a_{ij}]$ with complex number entries satisfying

(1) $$\sum_i a_{ij} = \sum_j a_{ij} = \sum_i a_{ii} = \sum_i a_{i\,n-i+1}$$

Definition. Call A (multiplicatively) balanced if

(2) $$\sum_j \prod_i a_{ij} = \sum_i \prod_j a_{ij} ,$$

and completely balanced if

(3) $$\sum_j \prod_i (a_{ij} + z = \sum_i \prod_j (a_{ij} + z) \quad ,$$

for all complex number z.

These two properties are explored for $n = 3, 4$ and 5. Note that magic squares are our main object and there are millions of them which satisfy (1), of order 5 alone.

2. THEOREM

These squares of order 3 are all completely balanced.

Proof. It is well known (see [2]) that (1) implies

$$[a_{ij}] = \begin{bmatrix} k+a & k-a-b & k+b \\ k-a+b & k & k+a-b \\ k-b & k+a+b & k-a \end{bmatrix} ,$$

where k, a, b are arbitrary parameters.

A direct computation can show (2). An easy way to see this is to change (2) into a determinant as follows:

$$\sum_j \prod_i a_{ij} - \sum_i \prod_j a_{ij} = \begin{vmatrix} a_{11} & a_{22} & a_{33} \\ a_{23} & a_{31} & a_{12} \\ a_{32} & a_{13} & a_{21} \end{vmatrix} = \begin{vmatrix} k+a & k & k-a \\ k+a-b & k-b & k-a-b \\ k+a+b & k+b & k-a+b \end{vmatrix} = 0$$

because the first row is the average of the other two rows.

However, the majority of magic squares of order $n(> 3)$ are not balanced. For example, the famous Dürer's magic square (Fig. 1) is not balanced and the second one (Fig. 2) is balanced and also completely.

An $n \times n$ matrix A, to be completely balanced, all the coefficients of the polynomial in z, say

$$\sum_i c_i z^i ,$$

obtained from (3) have to be 0. Equation (2) is merely $c_0 = 0$. If $c_0 = 0$, i.e., A is balanced, to determine whether A

$$\begin{bmatrix} 16 & 3 & 2 & 13 \\ 5 & 10 & 11 & 8 \\ 9 & 6 & 7 & 12 \\ 4 & 15 & 14 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 14 & 7 & 12 \\ 15 & 4 & 9 & 6 \\ 10 & 5 & 16 & 3 \\ 8 & 11 & 2 & 13 \end{bmatrix}$$

c.p.s. = 8,984
r.p.s. = 11,024
c.p.s. for column-product sum

Figure 1

p.s. = 9,104

Figure 2

is further completely balanced it is sufficient to show, by the fundamental theorem of algebra, that the above polynomial is satisfied by any n different values of z. In fact, checking for $n-4$ $(n>3)$ values of z is enough. For: $c_n = n - n = 0$,

$$c_{n-1} = \sum_j \sum_i a_{ij} - \sum_i \sum_j a_{ij} = 0,$$

$$c_{n-2} = \sum_j \sum_{i<k} a_{ij}a_{kj} - \sum_j \sum_{i<k} a_{ji}a_{jk} = \frac{1}{2}\left[\sum_j \sum_{i\neq k} a_{ij}a_{kj} - \sum_j \sum_{i\neq k} a_{ji}a_{jk}\right]$$

$$= \frac{1}{2}\left[\sum_{i,j} a_{ij} \sum_{k\neq i} a_{kj} - \sum_{i,j} a_{ji} \sum_{k\neq i} a_{jk}\right] = \frac{1}{2}\left[\sum_{i,j} a_{ij}(S-a_{ij}) - \sum_{i,j} a_{ji}(S-a_{ji})\right]$$

$$= \frac{1}{2}\left[S\sum_{i,j} a_{ij} - \sum_{i,j} a_{ij}^2 - S\sum_{i,j} a_{ji} + \sum_{i,j} a_{ji}^2\right] = 0,$$

where S is the row (or column) sum, and

$$c_{n-3} = \sum_t \left[\sum_{i<j<k} a_{it}a_{jt}a_{kt} - \sum_{i<j<k} a_{ti}a_{tj}a_{tk}\right] = \frac{1}{6}\sum_t \left[\sum_{i\neq j} a_{it}a_{jt}(S-a_{it}-a_{jt}) - \sum_{i\neq j} a_{ti}a_{tj}(S-a_{ti}-a_{tj})\right]$$

$$= \frac{1}{6}\sum_t \left[S\sum_{i\neq j} a_{it}a_{jt} - 2\sum_{i\neq j} a_{it}^2 a_{jt} - S\sum_{i\neq j} a_{ti}a_{tj} + 2\sum_{i\neq j} a_{ti}^2 a_{tj}\right]$$

$$= \frac{1}{6}\sum_t \left[S\sum_{i\neq j} (a_{it}a_{jt} - a_{ti}a_{tj}) - 2\sum_i a_{it}^2(S-a_{it}) + 2\sum_i a_{ti}^2(S-a_{ti})\right]$$

(the first sum is 0 as in c_{n-2})

$$= \frac{1}{3}\sum_t \left[S\sum_i (a_{ti}^2 - a_{it}^2) + \sum_i (a_{it}^3 - a_{ti}^3)\right] = \frac{1}{3}\left[S\sum_{t,i} (a_{ti}^2 - a_{it}^2) + \sum_{t,i} (a_{it}^3 - a_{ti}^3)\right]$$

$$= 0 \, .$$

The above fact implies the following.

Theorem. Any balanced square of order 4 is completely balanced.

For $n(>4)$ we are unable to show $c_{n-4} = 0$. An obstruction is the appearance of the sum

$$\sum_t \left(\sum_{i \neq j} a_{it}^2 a_{jt}^2 - \sum_{i \neq j} a_{ti}^2 a_{tj}^2 \right)$$

in c_{n-4}. Since

$$2 \sum_{i \neq j} a_{it}^2 a_{jt}^2 = \left(\sum_i a_{it}^2 \right)^2 - \sum_i a_{it}^4 ,$$

a sufficient condition for $c_{n-4} = 0$ or a condition that any balanced square of order 5 to be completely balanced may be stated by

$$\sum_t \left(\sum_i a_{it}^2 \right)^2 = \sum_t \left(\sum_i a_{ti}^2 \right)^2 . \tag{4}$$

Incidentally, Eq. (4) is the condition easily satisfied by any doubly magic square, a magic square $[a_{ij}]$ such that $[a_{ij}^2]$ is also a magic square. Summarizing the above argument we state a theorem.

Theorem. If a balanced square of order 5 satisfies the condition (4), then it is completely balanced.

In the theorem (4) is a sufficient condition and we do not know whether it is necessary. All the balanced magic squares of order 5 that we have been able to check turned out to be also completely balanced and they do satisfy (4). Thus, we make a conjecture.

Conjecture. A balanced magic square of order 5 is completely balanced.

3. CONSTRUCTION OF BALANCED SQUARES

Some magic squares of order 4 or 5 constructed by adding two orthogonal Latin squares seem balanced (also completely). For example:

$$\begin{bmatrix} a & d & b & c \\ d & a & c & b \\ c & b & d & a \\ b & c & a & d \end{bmatrix} + \begin{bmatrix} u & v & x & y \\ x & y & u & v \\ v & u & y & x \\ y & x & v & u \end{bmatrix} = \begin{bmatrix} 1 & 4 & 2 & 3 \\ 4 & 1 & 3 & 2 \\ 3 & 2 & 4 & 1 \\ 2 & 3 & 1 & 4 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 5 & 10 & 20 \\ 10 & 20 & 0 & 5 \\ 5 & 0 & 20 & 10 \\ 20 & 10 & 5 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 9 & 12 & 23 \\ 14 & 21 & 3 & 7 \\ 8 & 2 & 24 & 11 \\ 22 & 13 & 6 & 4 \end{bmatrix}$$

$p.s. = 19{,}646$

$$\begin{bmatrix} a & b & c & d & e \\ d & e & a & b & c \\ b & c & d & e & a \\ e & a & b & c & d \\ c & d & e & a & b \end{bmatrix} + \begin{bmatrix} x & y & s & t & v \\ s & t & v & x & y \\ v & x & y & s & t \\ y & s & t & v & x \\ t & v & x & y & s \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 1 & 2 & 3 \\ 2 & 3 & 4 & 5 & 1 \\ 5 & 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 1 & 2 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 5 & 10 & 15 & 20 \\ 10 & 15 & 20 & 0 & 5 \\ 20 & 0 & 5 & 10 & 15 \\ 5 & 10 & 15 & 20 & 0 \\ 15 & 20 & 0 & 5 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 7 & 13 & 19 & 25 \\ 14 & 20 & 21 & 2 & 8 \\ 22 & 3 & 9 & 15 & 16 \\ 10 & 11 & 17 & 23 & 4 \\ 18 & 24 & 5 & 6 & 12 \end{bmatrix}$$

$p.s. = 607{,}425$ diagonal $p.s. = 599{,}399$

Figure 3

REMARKS

1. We do not know any nontrivial (all different entries) balanced square of order greater than 5. We constructed a magic square of order 10 from the famous pair of orthogonal Latin squares of that order, but we found it not balanced.
2. We do not know an example of a balanced magic square which is not completely balanced.
3. Magic squares of order 6, 7 and 8 appearing in Andrews' book [1] are not balanced.
4. We did not encounter yet a balanced square whose two-way diagonal product sums are equal to the row product sum (really diabolic one) but at least two diagonal product sums alone can be equal as in Fig. 3.

REFERENCES

1. W.S. Andrews, *Magic Squares and Cubes,* Dover, 1960.
2. Jack Chernick, "Solution of the General Magic Square," *Math. Monthly,* March 1938, pp. 172–175.

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Likewise, it is obvious by inspection of a table of Fibonacci primes ($\geqslant 5$) that they are $\equiv 1 \pmod 4$ and thus expressable as the sum of the square of two smaller integers; specifically, it is well known that

$$U_p = U^2_{(p-1)/2} + U^2_{\frac{(p-1)}{2}+1}$$

where U_p is a Fibonacci prime ($\geqslant 5$).

Thus, it is perceived that the Mersenne and Fibonacci primes ($\geqslant 5$) form two mutually exclusive sets; i.e., *no* primes ($\geqslant 5$) can be both a Mersenne and a Fibonacci prime.

REFERENCE

1. William Raymond Griffin, "Mersenne Primes—The Last Three Digits," *J. Recreational Math,* 5 (1), p. 53, Jan., 1972.

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