ON FIBONACCI NUMBERS OF THE FORM $k^2 + 1$

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Consider the Diophantine equation

(1) $$(X - Y)^7 = X^5 - Y^5,$$

where X, Y are to be integers. We have an infinitude of trivial solutions of (1) given by $X = m$, $Y = m$, where m is an integer parameter. We shall concern ourselves here with solutions (X,Y) of (1) for which $X \neq Y$. There is no loss of generality in assuming that $X > Y$.

Using an idea of Rotkiewicz (cf. Sierpiński [5]), we let $d = (X,Y)$ and $X = dx$, $Y = dy$. Substituting this in (1) and rearranging terms, we get

$$d^2(x-y)^6 = (x-y)^4 + 5xy(x-y)^2 + 5x^2y^2 .$$

Since $(x,y) = 1$, $x > y$, and $(x - y)$ must divide $5x^2y^2$, we must have $x - y = 1$. Hence

(2) $$d^2 = 5y^4 + 10y^3 + 10y^2 + 5y + 1 .$$

We rewrite (2) as

$$16d^2 = 5[(2y+1)^2 + 1]^2 - 4 .$$

Putting

$$v = 2d \qquad \text{and} \qquad u = [(2y+1)^2 + 1]/2 ,$$

we have the familiar equation

(3) $$v^2 - 5u^2 = -1 .$$

Now it is well known that if (v,u) is any solution of (3), there exists an integer m such that

$$u = (F_{6m+3})/2 ,$$

where the Fibonacci numbers F_n $(|n| = 0, 1, 2, \cdots)$ are defined by the recurrence relation

$$F_{n+1} = F_n + F_{n-1} \qquad (|n| = 0, 1, 2, \cdots)$$

together with the initial conditions $F_0 = 0$, $F_1 = 1$. Thus, in order for (1) to have a solution, we must have an integer m such that

$$F_{6m+3} = (2y+1)^2 + 1 .$$

In Gryte *et al.* [3] it was shown, by means of a computer search, that the more general equation

(4) $$F_n = k^2 + 1$$

has no solution for any n such that $5 < n \leqslant 10^6$. In this note we will show that all solutions of (4) are given by $n = \pm 1, 2, \pm 3, \pm 5$. Hence, the only solutions of (1) such that $X > Y$ are (1,0) and (0, −1).

We first note that since $3 \mid F_n$ if and only if $4 \mid n$, (4) has no solution if $4 \mid n$. From Lucas' [4] identities (52), we see that

$$F_{2m+1} - 1 = F_m L_{m+1} \quad \text{when} \quad 2 \mid m,$$
$$F_{2m+1} - 1 = F_{m+1} L_m \quad \text{when} \quad 2 \nmid m,$$
$$F_{2m} - 1 = F_{m-1} L_{m+1} \quad \text{when} \quad 2 \nmid m .$$

Here L_n $(|n| = 0, 1, 2, \cdots)$ are the Lucas numbers defined by

$$L_{n+1} = L_n + L_{n-1} \qquad (|n| = 0, 1, 2, \cdots)$$

together with $L_0 = 2$, $L_1 = 1$. We also have

$$2L_{m+1} = L_m + 5F_m = 3L_{m-1} + 5F_{m-1},$$

$$2F_{m+1} = L_m + F_m.$$

If p is any prime divisor of F_m and L_{m+1}, then p is a prime divisor of L_m. Since $(F_m, L_m) = 1,2$, we see that p must be 2. From the fact that $2\nmid(L_m, L_{m+1})$, it follows that $(F_m, L_{m+1}) = 1$. Using similar reasoning, it is not difficult to show that $(F_{m+1}, L_m) = 1$. Finally, if $p\,|\,(L_{m+1}, F_{m-1})$ and $2\nmid m$, then $p\,|\,3L_{m-1}$. In this case it is possible for $p = 3$. If $p \neq 3$, then $p\,|\,(L_{m-1}, F_{m-1})$ and $p = 2$, but, since $2\nmid(L_m, L_{m+1})$, this is not possible. If $9\,|\,(L_{m+1}, F_{m-1})$, then $3\,|\,L_{m-1}$, which is also impossible; consequently, $(L_{m+1}, F_{m-1}) = 1$ or 3.

In order to solve (4) we consider two cases.

Case (i). n odd.

Here we have

$$k^2 = F_{(n-1)/2}L_{(n+1)/2} \quad \text{or} \quad k^2 = F_{(n+1)/2}L_{(n-1)/2}.$$

In either event, we must have some integer $r = (n \pm 1)/2$ such that $|F_r|$ is an integer square. The only possible values for r are ± 1, 0, ± 2, ± 12 (see Wyler [6] or Cohn [1]); hence, it is a simple matter to discover that the only solutions of (4) for odd n are $n = \pm 1, \pm 3, \pm 5$.

Case (ii). n even.

In this case $4\nmid n$ and

$$k^2 = F_{n/2-1}L_{n/2-1}.$$

If $(F_{n/2-1}, L_{n/2+1}) = 1$, we have

$$F_{n/2-1} = t^2 \quad \text{and} \quad n/2 - 1 = \pm 1, 0, 2, 12.$$

The only possible value of n such that (4) is satisfied is $n = 2$. If $(F_{n/2-1}, L_{n/2+1}) = 3$, we have $F_{n/2-1} = 3s^2$ for some integer s. Putting $r = L_{n/2-1}$ and noting that $n/2 - 1$ is even, we see from the identity

$$L_m^2 - 5F_m^2 = 4(-1)^m$$

that

$$r^2 - 45s^4 = 4. \tag{5}$$

Since the Diophantine equation

$$x^2 - 45y^2 = 4$$

has the fundamental solution $x = 7$, $y = 1$ and the equation

$$x^2 - 45y^2 = -4$$

has no integer solution, we see from Cohn [2] that the only possible solutions of (5) are given by

$$s^2 = 0, u_1, u_2, u_3,$$

where $u_1 = 1$, $u_2 = 7$, $u_3 = 48$. That is, the only solutions of (5) are $(\pm 2, 0)$, $(\pm 7, \pm 1)$. It follows that $F_{n/2-1} = 0, 3$ and the only possible even value of n such that (4) is satisfied is $n = 2$.

REFERENCES

1. J.H.E. Cohn, "On Square Fibonacci Numbers," *J. London Math. Soc.,* Vol. 39 (1964), pp. 537–540.
2. J.H.E. Cohn, "Five Diophantine Equations," *Math. Scand.,* Vol. 21 (1967), pp. 61–70.
3. D.G. Gryte, R.A. Kingsley, and H.C. Williams, "On Certain Forms of Fibonacci Numbers," *Proc. of the Second Louisiana Conference on Combinatorics, Graph Theory, and Computing,* Baton Rouge, 1971, pp. 339–344.
4. Ed. Lucas, "Théorie des Fonctions numériques Simplement périodiques," *Amer. J. of Math.,* Vol. 1 (1878), pp. 184–240, 289–321.
5. W. Sierpiński, *Elementary Theory of Numbers,* Panstwowe Wydawnictwo Naukowe, Warsaw (1964), p. 94.
6. O. Wyler, "Solutions of the problem. In the Fibonacci series $F_1 = 1$, $F_2 = 1$, $F_{n+1} = F_n + F_{n-1}$, the First, Second and Twelfth Terms are Squares. Are There any Offers?", *Amer. Math. Monthly,* Vol. 71 (1964), pp. 220–222.

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