# THE RANK OF APPARITION OF A GENERALIZED FIBONACCI SEQUENCE

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## 1. INTRODUCTION

In [4] Waddill and Sacks discuss a generalized Fibonacci sequence $\{K_n\}$, where $K_0 = 0, K_1 = 1, K_2 = 1$, and

$$K_{n+1} = K_n + K_{n-1} + K_{n-2}.$$

Several other properties of this sequence, often called the Tribonacci Sequence, may be easily deduced from the more general results of Miles [2] and Williams [5].

We give here the definition of the rank of apparition of an integer $m$ in the sequence $\{K_n\}$.

**Definition.** The rank of apparition of an integer $m$ in the sequence $\{K_n\}$ is the least positive integer $\rho$ for which

$$K_{\rho-1} \equiv K_\rho \equiv 0 \pmod m.$$

This definition is analogous to that for the ordinary Fibonacci sequence (see, for example, Vinson [3]). In [5] it was shown that such a rank of apparition always exists for any integer $m$; the purpose of this note is to determine, more precisely than was done in [5], the rank of apparition of any prime $p$.

## 2. PRELIMINARY RESULTS

We shall require a theorem of Cailler [1], which we only state here.

*Theorem.* Let $R,S$ be given integers and let $p(>3)$ be a prime such that $(p,R) = 1$. Let $\Delta = 4R^3 + 27S^2$ and put $q$ equal to the value of the Legendre symbol $(3\Delta | p)$.

If $p \equiv -q \pmod 3$, there is only one root in $GF[p]$ of

$$x^3 + Rx + S \equiv 0 \pmod p. \tag{2.1}$$

If $p \equiv q \pmod 3$, put $m = (p-q)/3$. There are three roots of (2.1) in $GF[p]$ if

$$U_m \equiv 0 \pmod p. \tag{2.2}$$

If (2.2) is not satisfied, there are no roots of (2.1) in $GF[p]$. Here $U_n$ is the Lucas Function defined by the recurrence relation

$$U_{n+1} = PU_n - QU_{n-1}$$

and the initial conditions $U_0 = 0, U_1 = 1$. $P$ and $Q$ are determined from the relations

$$3Q \equiv -R, \quad RP \equiv -3S \pmod p.$$

We also require the following

*Theorem.* (Williams [5]). If $K_{n-1} \equiv K_n \equiv 0$ (mod m) and $\rho$ is the rank of apparition of $m$, then $\rho$ is a divisor of $n$.

Finally, we need the fact [5] that

$$K_n = \frac{1}{D}\begin{vmatrix} 1 & \alpha & \alpha^{n+1} \\ 1 & \beta & \beta^{n+1} \\ 1 & \gamma & \gamma^{n+1} \end{vmatrix}, \tag{2.3}$$

where $\alpha, \beta, \gamma$ are the three roots of

$$x^3 - x^2 - x - 1 = 0$$

and $D$ is the value of the Vandermonde determinant

$$\begin{vmatrix} 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \\ 1 & \gamma & \gamma^2 \end{vmatrix}.$$

## 3. THE MAIN RESULT

Let $F(x)$ be the polynomial $x^3 - x^2 - x - 1$. If $F(x)$ is irreducible modulo $p$, let $G = GF[p^3]$ be the splitting field of $F(x) \pmod p$ and let $\theta, \phi = \theta^p, \psi = \theta^{p^2}$ be the roots of

(3.1) $$F(x) = 0$$

in $G$. Then in $G$ we have

$$\theta\phi\psi = 1 = \theta^{1+p+p^2} = \phi^{1+p+p^2} = \psi^{1+p+p^2}.$$

From (2.3) we have

$$K_{p^2+p} = K_{p^2+p+1} = 0.$$

If $p \equiv 1 \pmod 3$,

$$\theta^{(p^2+p+1)(p-1)/3} = 1;$$

hence,

$$\theta^{(p^2+p+1)/3} = \theta^{p(p^2+p+1)/3} = \phi^{(p^2+p+1)/3} = \psi^{(p^2+p+1)/3}$$

and

$$K_{(p^2+p-2)/3} = K_{(p^2+p+1)/3} = 0.$$

If $F(x)$ is factorable modulo $p$ into a linear and irreducible quadratic factor, let $G = GF[p^2]$ be the splitting field of $F(x)$ and let $\theta \in GF[p]$, $\phi$, $\psi = \phi^p$ be the roots of (3.1) in $G$. If $p \equiv 1 \pmod 3$,

$$\phi^{(p^2-1)/3}\theta^{(p-1)/3} = 1;$$

thus,

$$\theta^{(p^2-1)/3} = \phi^{(p^2-1)/3} = \psi^{(p^2-1)/3}$$

and

(3.2) $$K_{(p^2-4)/3} = K_{(p^2-1)/3} = 0.$$

If $p \equiv -1 \pmod 3$, we use the simple fact that

(3.3) $$x^2(x-1)^3 = 4,$$

if $F(x) = 0$. Hence, in $G$

$$(\phi^2(\phi-1)^3)^{(p^2-1)/3} = 4^{(p^2-1)/3}$$

and

$$\phi^{(p^2-1)/3} = \theta^{(p^2-1)/3} = \psi^{(p^2-1)/3}.$$

We again have (3.2).

If $F(x)$ is factorable modulo $p$ into three linear factors, let $\theta, \phi, \psi \in GF[p]$ be the roots of (3.1). We have

$$\theta^{p-1} \equiv \phi^{p-1} \equiv \psi^{p-1} \equiv 1 \pmod p$$

and

$$K_{p-2} \equiv K_{p-1} \equiv 0 \pmod p.$$

If $p \equiv 1 \pmod 3$, from (3.3)

$$\theta^{2(p-1)/3} \equiv 4^{(p-1)/3} \equiv \phi^{2(p-1)/3} \equiv \psi^{2(p-1)/3} \pmod p;$$

hence, we have

$$\theta^{(p-1)/3} \equiv \phi^{(p-1)/3} \equiv \psi^{(p-1)/3} \pmod p$$

and

$$K_{(p-4)/3} \equiv K_{(p-1)/3} \equiv 0 \pmod p.$$

Since

$$(-6)^3F(x) \equiv (-6x+2)^3 + 48(-6x+2) + 304,$$

we can put together the above results and the theorems of Section 2 to obtain the following

*Theorem.* (The law of apparition for the Tribonacci sequence). Let $U_n$ be defined by the linear recurrence

$$U_{n+1} = 19U_n - 16U_{n-1}$$

and the initial values $U_0 = 0$, $U_1 = 1$.

If $p$ is a prime ($\neq$ 2,3,11) and $p \equiv -(33|p)$ (mod 3), the rank of apparition $\rho$ of $p$ is a divisor of $(p^2 - 1)/3$. If $p \equiv (33|p) \equiv 1 \pmod 3$, $\rho$ is a divisor of $(p-1)/3$ when $p$ divides $U_{(p-1)/3}$; if $p$ does not divide $U_{(p-1)/3}$, $\rho$ is a divisor of $(p^2 + p + 1)/3$. If $p \equiv (33|p) \equiv -1 \pmod 3$, $\rho$ is a divisor of $p - 1$ when $U_{(p+1)/3}$ is divisible by $p$; if $p$ does not divide $U_{(p+1)/3}$, $\rho$ is a divisor of $p^2 + p + 1$. If $p = 2$, $\rho = 4$; if $p = 3$, $\rho = 13$; and, if $p = 11$, $\rho = 110$.

The last results were obtained by direct calculation.

## 4. TABLE

We give here a table of values of $p$ and $\rho$ for all $p \leqslant 347$.

| $p$ | $\rho$ | $p$ | $\rho$ | $p$ | $\rho$ | $p$ | $\rho$ |
|---|---|---|---|---|---|---|---|
| 2 | 4 | 67 | 1519 | 157 | 8269 | 257 | 256 |
| 3 | 13 | 71 | 5113 | 163 | 54 | 263 | 23056 |
| 5 | 31 | 73 | 1776 | 167 | 9296 | 269 | 268 |
| 7 | 16 | 79 | 1040 | 173 | 2494 | 271 | 24480 |
| 11 | 110 | 83 | 287 | 179 | 32221 | 277 | 12788 |
| 13 | 56 | 89 | 8011 | 181 | 10981 | 281 | 13160 |
| 17 | 96 | 97 | 3169 | 191 | 36673 | 283 | 13348 |
| 19 | 120 | 101 | 680 | 193 | 1552 | 293 | 28616 |
| 23 | 553 | 103 | 17 | 197 | 3234 | 307 | 10472 |
| 29 | 140 | 107 | 1272 | 199 | 66 | 311 | 310 |
| 31 | 331 | 109 | 330 | 211 | 1855 | 313 | 32761 |
| 37 | 469 | 113 | 12883 | 223 | 16651 | 317 | 100807 |
| 41 | 560 | 127 | 1792 | 227 | 17176 | 331 | 36631 |
| 43 | 308 | 131 | 5720 | 229 | 17557 | 337 | 5408 |
| 47 | 46 | 137 | 18907 | 233 | 9048 | 347 | 40136 |

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