SUMS OF PRODUCTS OF GENERALIZED FIBONACCI NUMBERS

GEORGE BERZSENYI
Lamar University, Beaumont, Texas 77705

The purpose of this note is to announce the following formulae, where H_0 and H_1 are chosen arbitrarily and $H_n = H_{n-1} + H_{n-2}$ for $n > 1$:

$$(*)\qquad \sum_{k=0}^{n} H_k H_{k+2m+1} = \begin{cases} H^2_{m+n+1} - H^2_{m+1} + H_0 H_{2m+1}, & \text{if } n \text{ is even} \\ H^2_{m+n+1} - H^2_m, & \text{if } n \text{ is odd} \end{cases}$$

$$\sum_{k=0}^{n} H_k H_{k+2m} = \begin{cases} H_{m+n} H_{m+n+1} - H_m H_{m+1} + H_0 H_{2m}, & \text{if } n \text{ is even} \\ H_{m+n} H_{m+n+1} - H_{m-1} H_m, & \text{if } n \text{ is odd} . \end{cases}$$

These results may be established by first proving the corresponding formulas for Fibonacci numbers and then expanding the expressions on the left side of (*) by using the well-known relation

$$H_n = F_{n-1} H_0 + F_n H_1 .$$

To prove (*) for Fibonacci numbers the method of generating functions is utilized. Using Binet's formulae for Fibonacci and Lucas numbers, one finds that

$$\sum_{n=0}^{\infty} F^2_{n+m} x^n = \frac{F^2_m + [F_{m-1}F_m + (-1)^m]x - F^2_{m-1}x^2}{(1+x)(1-3x+x^2)} \quad \text{and} \quad \sum_{n=0}^{\infty} F_n F_{n+m} x^n = \frac{F_{m+1}x - F_{m-1}x^2}{(1+x)(1-3x+x^2)} .$$

Moreover,

$$\sum_{n=0}^{\infty} \left(\sum_{n=0}^{n} F_k F_{k+m} \right) x^n = \left(\sum_{n=0}^{\infty} x^n \right) \left(\sum_{n=0}^{\infty} F_n F_{n+m} x^n \right) = \frac{F_{m+1}x - F_{m-1}x^2}{(1-x)(1+x)(1-3x+x^2)} ,$$

and with the methods of Gould [1] one can derive the bisection generating functions

$$\sum_{n=0}^{\infty} F^2_{2n+m} x^n = \frac{F^2_m + [(-1)^m - 3F_{m-2}F_m]x + F^2_{m-2}x^2}{(1-x)(1-7x+x^2)} ,$$

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^{2n} F_k F_{k+m} \right) x^n = \frac{F_{m+3}x - F_{m-1}x^2}{(1-x)(1-7x+x^2)} ,$$

and

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^{2n+1} F_k F_{k+m} \right) x^n = \frac{F_{m+1} - F_{m-3}x}{(1-x)(1-7x+x^2)} .$$

The proof of (*) for Fibonacci numbers is then completed by observing the relationships among these generating functions. For example,

$$\sum_{n=0}^{\infty} \left(F_{2n+m+2}^2 - F_m^2 \right) x^n = \sum_{n=0}^{\infty} F_{2n+m+2}^2 x^n - F_m^2 \sum_{n=0}^{\infty} x^n$$

$$= \frac{F_{m+2}^2 + [(-1)^{m+2} - 3F_m F_{m+2}]x + F_m^2 x^2}{(1-x)(1-7x+x^2)} - \frac{F_m^2}{1-x}$$

$$= \frac{(F_{m+2}^2 - F_m^2) + [(-1)^m - 3F_m F_{m+2} + 7F_m^2]x}{(1-x)(1-7x+x^2)}$$

$$= \frac{F_{2m+2} - F_{2m-2}x}{(1-x)(1-7x+x^2)}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{2n+1} F_k F_{k+2m+1} \right) x^n ,$$

and hence,

$$\sum_{k=0}^{2n+1} F_k F_{k+2n+1} = F_{2n+m+2}^2 - F_m^2 .$$

The other three cases are similar.

REFERENCE

1. V. E. Hoggatt, Jr., and J. C. Anaya, "A Primer for the Fibonacci Numbers: Part XI," *The Fibonacci Quarterly,* Vol. 11, No. 1 (Feb., 1973), pp. 85–90.

****

[Continued from P. 342.]

Proof. The corollary is known to be true for $(b/-1) = 1$. Then the following results can be calculated:

If

$$(a_1 a_2/-1) = 1,$$

then

$$(a_1 a_2/b) = 1,$$
$$(-a_1 a_2/b) = (-1/b),$$
$$(a_1 a_2/-b) = 1,$$
$$(-a_1 a_2/-b) = -(-1/b);$$

If $(a_1 a_2/-1) = -1$, then

$$(a_1 a_2/b) = 1,$$
$$(-a_1 a_2/b) = (-1/b),$$
$$(a_1 a_2/-b) = -1,$$
$$(-a_1 a_2/-b) = (-1/b).$$

[Continued on P. 349.]