A PRIMER ON THE PELL SEQUENCE AND RELATED SEQUENCES

MARJORIE BICKNELL
Wilcox High School, Santa Clara, California 95051

1. INTRODUCTION

Regular readers of this journal are well acquainted with basic properties and identities relating to the Fibonacci sequence and its associated sequence, the Lucas sequence, but may be unaware that the Pell sequence is one of many other sequences which share a large number of the same basic properties. The reader should supply the analogous Fibonacci identities, verify formulas numerically, and provide proofs for formulae given here. The proofs are very similar to those for the Fibonacci case.

2. THE PELL SEQUENCE

By observation of the sequence $\{1, 2, 5, 12, 29, 70, 169, \cdots, P_n, \cdots\}$ it is easily seen that each term is given by

(1) $$P_n = 2P_{n-1} + P_{n-2}, \qquad P_1 = 1, \qquad P_2 = 2.$$

The sequence can be extended to include

$$P_0 = 0, \qquad P_{-1} = 1, \qquad P_{-2} = -2, \qquad P_{-3} = 5, \quad \cdots \quad P_{-n} = (-1)^{n+1}P_n .$$

The associated sequence $\{R_n\}$, where $R_n = P_{n-1} + P_{n+1}$, has

(2) $$R_n = 2R_{n-1} + R_{n-2}, \qquad R_1 = 2, \qquad R_2 = 6,$$

with first few members given by 2, 6, 14, 34, 82, 198, ⋯, and can be extended to include

$$R_0 = 2, \qquad R_{-1} = -2, \qquad \cdots, \qquad R_{-n} = (-1)^n R_n .$$

The Pell numbers enjoy a Binet form. If we take the equation

$$y^2 = 2y + 1$$

which has roots

$$\alpha = (2 + \sqrt{8})/2 \qquad \text{and} \qquad \beta = (2 - \sqrt{8})/2,$$

then it can be proved by mathematical induction that

(3) $$P_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad R_n = \alpha^n + \beta^n, \qquad \alpha^n = \frac{R_n + P_n\sqrt{8}}{2} \quad .$$

Using the Binet form, one can prove that P_{nk} is evenly divisible by P_k, $k \neq 0$, so that the Pell sequence also shares many divisibility properties of the Fibonacci numbers.

Geometrically, the Fibonacci numbers are related to the Golden Rectangle, which, of course, has the property that upon removing one square with edge equal to the width of the rectangle, the rectangle remaining is again a Golden Rectangle. The equation related to the Pell numbers arises from the ratio of length to width in a "silver rectangle" of length y and width 1 such that, when two squares with side equal to the width are removed, the remaining rectangle has the same ratio of length to width as did the original rectangle, or such that

$$\frac{y}{1} = \frac{1}{y - 2}, \qquad y^2 - 2y - 1 = 0,$$

so that $y = \alpha$, the positive root given above.

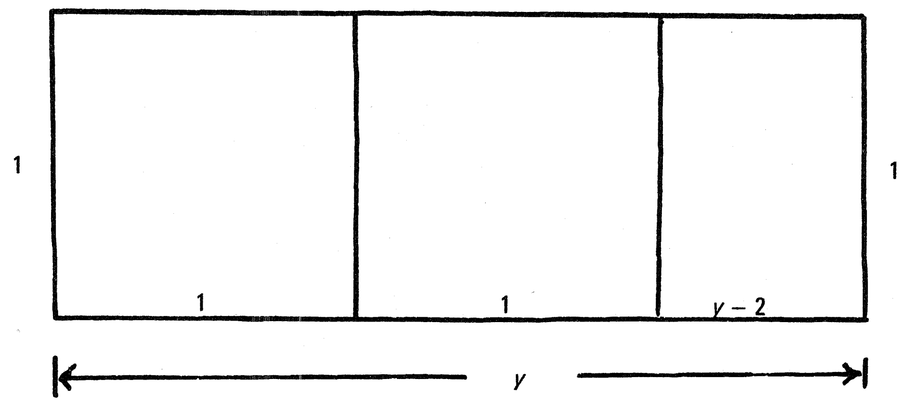

Figure 1

Some simple identities for Pell numbers follow. No attempt was made for completeness; these identities merely indicate some directions that can be explored in finding identities. (Most of these identities can be found in Serkland [1] and Horadam [2].)

(4) $$P_{n+1}P_{n-1} - P_n^2 = (-1)^n$$

(5) $$R_{n+1}R_{n-1} - R_n^2 = 8(-1)^{n+1}$$

(6) $$R_n = P_{n+1} + P_{n-1}$$

(7) $$8P_n = R_{n+1} + R_{n-1}$$

(8) $$P_{2n+1} = P_{n+1}^2 + p_n^2$$

(9) $$2P_{2n} = P_{n+1}^2 - P_{n-1}^2$$

(10) $$P_{-n} = (-1)^{n+1}P_n$$

(11) $$R_{-n} = (-1)^n R_n$$

(12) $$P_{2n} = P_n R_n$$

(13) $$R_n + R_{n+1} = 4P_{n+1}$$

(14) $$R_0 + R_1 + R_2 + \cdots + R_n = 2P_{n+1}$$

(15) $$P_{n+p+1} = P_{n+1}P_{p+1} + P_n P_p$$

(16) $$R_n^2 - 8P_n^2 = 4(-1)^n$$

(17) $$P_1^2 + P_2^2 + P_3^2 + \cdots + P_n^2 = (P_n P_{n+1})/2$$

(18) $$\frac{x}{1 - 2x - x^2} = \sum_{i=1}^{\infty} P_n x^n$$

(19) $$\sum_{k=0}^{n} \binom{n}{k} 2^k P_{2k} = P_{2n}$$

(20) $$\sum_{k=0}^{n} \binom{n}{k} P_k P_{n-k} = 2^n P_n$$

The Fibonacci numbers were generated by a matrix

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

which satisfied the equation whose roots provide the Binet form for the Fibonacci numbers. The Pell numbers are also generated by a matrix M,

$$M = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}, \qquad M^n = \begin{pmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{pmatrix}$$

which can be proved by mathematical induction. The matrix M provides some identities immediately. For example,

$$\det M^n = (\det M)^n = (-1)^n = P_{n+1}P_{n-1} - P_n^2$$

and expanding $M^{n+p} = M^n M^p$ gives

$$P_{n+p+1} = P_{n+1}P_{p+1} + P_nP_p$$

upon equating elements in the upper left. The matrix M also satisfies the equation related to the Pell sequence, $M^2 = 2M + 1$.

3. THE GENERAL SEQUENCE

Since the Fibonacci sequence and the Pell sequence share so many basic properties, and since they have the same starting values but different, though related, recurrence relations, it seems reasonable to ask what properties the sequence $\{U_n\}$,

(21) $$U_0 = 0, \qquad U_1 = 1, \qquad U_{n+1} = bU_n + U_{n-1},$$

which includes both the Fibonacci sequence ($b = 1$) and the Pell sequence ($b = 2$) as special cases, will have.

The first few values of $\{U_n\}$ are:

$$\begin{aligned} U_0 &= 0 \\ U_1 &= 1 \\ U_2 &= b \\ U_3 &= b^2 + 1 \\ U_4 &= b^3 + 2b \\ U_5 &= b^4 + 3b^2 + 1 \\ U_6 &= b^5 + 4b^3 + 3b \\ U_7 &= b^6 + 5b^4 + 6b^2 + 1 \\ U_8 &= b^7 + 6b^5 + 10b^3 + 4b \\ \cdots & \quad \cdots \end{aligned}$$

These are just the Fibonacci polynomials $F_n(x)$ (see [3]) given by

$$F_0(x) = 0, \qquad F_1(x) = 1, \qquad F_{n+1}(x) = xF_n(x) + F_{n-1}(x)$$

evaluated at $x = b$. That is,

$$F_n(1) = F_n, \qquad F_n(2) = P_n, \quad \text{and} \quad F_n(b) = U_n.$$

Thus, any known identities for Fibonacci polynomials establish the same identities for $\{F_n\}$, $\{P_n\}$, and $\{U_n\}$. $\{U_n\}$ has an associated sequence $\{V_n\}$, $V_n = U_{n-1} + U_{n+1}$, where

(22) $$V_0 = 2, \qquad V_1 = b, \qquad V_{n+2} = bV_{n+1} + V_n.$$

Using identities for Fibonacci polynomials given in [2], we have

(23) $$U_m = V_kU_{m-k} + (-1)^{k+1}U_{m-2k}$$

(24) $$U_{-n} = (-1)^{n+1}U_n$$

(25) $$V_{-n} = (-1)^nV_n$$

(26) $$V_n = bU_n + 2U_{n-1}$$

(27) $$bV_n = U_{n+2} - U_{n-2}$$

(28) $$U_{2n} = U_n V_n$$

(29) $$U_{m+n} + (-1)^n U_{m-n} = U_m V_n$$

Also from [2], we can also state that U_{nk} is always divided evenly by U_n, $n \neq 0$.

Now, if we explore the related equation

$$y^2 = by + 1$$

with roots

$$\alpha = \frac{b + \sqrt{b^2 + 4}}{2} \quad \text{and} \quad \beta = \frac{b - \sqrt{b^2+4}}{2},$$

it can be shown by mathematical induction that

(30) $$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad V_n = \alpha^n + \beta^n,$$

(31) $$\alpha^n = \frac{V_n + U_n\sqrt{b^2+4}}{2}.$$

Geometrically, U_n and V_n are related to "silver rectangles." (See Raab [4].) If a rectangle of length y and width 1 has dimensions such that, when b squares with side equal to the width are removed, the rectangle remaining has the same ratio of length to width as the original, then the ratio of length to width is $\alpha = (b + \sqrt{b^2+4})/2$, as seen by the following:

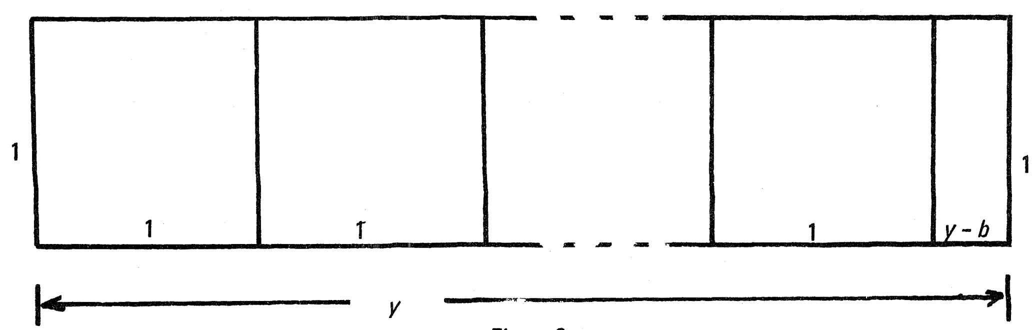

Figure 2

$$\frac{y}{1} = \frac{1}{y-b}, \qquad y^2 - by - 1 = 0, \qquad y = \frac{b+\sqrt{b^2+4}}{2}.$$

Further, it can be proved that

(32) $$\lim_{n \to \infty} \frac{U_{n+1}}{U_n} = \frac{b + \sqrt{b^2+4}}{2} = \alpha,$$

Serkland [1] and Horadam [2] establish that the generating function for $\{U_n\}$ is

(33) $$\frac{x}{1 - bx - x^2} = \sum_{i=0}^{\infty} U_n x^n.$$

Now, it is well known that the Fibonacci polynomials are generated by a matrix. (See [1], [3], for example.) That the matrix Q below generates $\{U_n\}$ can be established by mathematical induction:

$$Q = \begin{pmatrix} b & 1 \\ 1 & 0 \end{pmatrix}, \qquad Q^n = \begin{pmatrix} U_{n+1} & U_n \\ U_n & U_{n-1} \end{pmatrix} .$$

Since $\det Q^n = (\det Q)^n = (-1)^n$, we have

(34) $$U_{n+1}U_{n-1} - U_n^2 = (-1)^n .$$

Using $Q^{m+n} = Q^m Q^n$ and equating elements in the upper left gives us

(35) $$U_{m+n+1} = U_{m+1}U_{n+1} + U_m U_n$$

(36) $$U_{2n+1} = U_{n+1}^2 + U_n^2 .$$

Many other identities can be found in the same way. Note that the characteristic polynomial of Q is $x^2 - bx - 1 = 0$.

Summation identities can also be generalized [1], [2], as, for example,

(37) $$U_0 + U_1 + U_2 + \cdots + U_n = (U_n + U_{n+1} - 1)/b$$

(38) $$V_0 + V_1 + V_2 + \cdots + V_n = (V_n + V_{n+1} + b - 2)/b$$

(39) $$U_0^2 + U_1^2 + U_2^2 + \cdots + U_n^2 = (U_n U_{n+1})/b .$$

The reader is left to see what other identities he can find which hold for the general sequence.

REFERENCES

1. Carl E. Serkland, *The Pell Sequence and Some Generalizations,* Unpublished Master's Thesis, San Jose State University, San Jose, California, August, 1972.
2. A. F. Horadam, "Pell Identities," *The Fibonacci Quarterly,* Vol. 9, No. 3 (April 1971), pp. 245–252, 263.
3. Marjorie Bicknell, "A Primer for the Fibonacci Numbers: Part VII, An Introduction to Fibonacci Polynomials and Their Divisibility Properties," *The Fibonacci Quarterly,* Vol. 8, No. 4 (Oct. 1970), pp. 407–420.
4. Joseph A. Raab, "A Generalization of the Connection Between the Fibonacci Sequence and Pascal's Triangle," *The Fibonacci Quarterly,* Vol. 1, No. 3 (Oct. 1963), pp. 21–31.

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[Continued from P. 344.]

Corollary 2. If $ab \equiv 1 \pmod 2$ and $(a,b) = 1$, and if $b_1 \equiv b_2 \pmod{2a}$, then

$$(a/b_1b_2) = \left(\frac{(-1/a)}{(-1/b_1b_2)} \right) .$$

In other words,

$$(a/b_1b_2) = 1$$

if and only if $a \equiv 1 \pmod 4$ and/or $b_1b_2 \equiv 1 \pmod 4$.

Proof. From (b_1b_2/a), $(-b_1b_2/a)$, $(b_1b_2/-a)$ and $(-b_1b_2/-a)$, the following results can be obtained by quadratic reciprocity:

[Continued on P. 384.]