A RECURSIVE METHOD FOR COUNTING INTEGERS NOT REPRESENTABLE IN CERTAIN EXPANSIONS

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1. INTRODUCTION

Let $\{P_i\}_1^\infty$ be a sequence of positive integers satisfying the inequality

$$(1) \qquad P_{n+1} \geq 1 + \sum_1^n P_i \quad \text{for} \quad n \geq 1;$$

then it is well known ([1], Theorem 1; [2], Theorem 2; [3], Theorem 1) that any positive integer N possesses at most one representation as a sum of distinct terms from the sequence $\{P_i\}$. Such representations, when they exist, are thus unique, and we term a sequence $\{P_i\}$ of positive integers satisfying (1) a *sequence of uniqueness,* or briefly a *u-sequence.* Following Hoggatt and Peterson [1], we define $M(N)$ for each positive integer N as the number of positive integers less than N which are not representable as a sum of distinct terms from a given fixed u-sequence $\{P_i\}$. The principal result in [1] (Theorem 4) is that if N has a representation

$$N = \sum_1^n a_i P_i$$

with $\{a_i\}$ binary coefficients, then

$$M(N) = N - \sum_1^n a_i 2^{i-1},$$

so that an explicit formula for $M(N)$ is available for *representable* positive integers. In general, a closed form expression for $M(N)$ as a function of N does not exist; our purpose in the present paper is to derive an expression from which $M(N)$ may be readily calculated for an arbitrary positive integer N.

2. DERIVATION

Throughout the following analysis, $\{P_i\}_1^\infty$ will denote a fixed u-sequence; we wish to find a recursive algorithm for determining $M(N)$.

First, we recall ([1], Theorem 2) that

$$M(P_n) = P_n - 2^{n-1} \quad \text{for} \quad n \geq 1,$$

so that only values of N not coinciding with terms of the u-sequence need be considered.

Theorem 1. Let N be an integer satisfying $P_n < N < P_{n+1}$ for some $n \geq 1$.

$$\text{(i)} \qquad \text{If} \quad P_n < N \leq \sum_1^n P_i, \quad \text{then} \quad M(N) = M(P_n) + M(N - P_n).$$

(ii) If $\sum_1^n P_i < N < P_{n+1}$, then $M(N) = M\left(\sum_1^n P_i\right) + \left(N - \sum_1^n P_i\right) - 1 = N - 2^n$.

NOTE: Result (i) expresses $M(N)$ in terms of $M(P_n)$ and $M(N - P_n)$. But $N - P_n < P_n$ in case (i) since

$$P_n < N \leqslant \sum_1^n P_i \quad \text{implies} \quad 0 < N - P_n \leqslant \sum_1^{n-1} P_i < P_n,$$

the latter inequality following from the fact that $\{P_i\}$ is a u-sequence. Thus, if we consider the values $M(1), M(2), \cdots, M(P_n)$ as known, then $M(N)$ is determined from (i) whenever

$$P_n < N \leqslant \sum_1^n P_i,$$

while $M(N)$ is given explicitly by (ii) for the remaining values of N in (P_n, P_{n+1}).

Proof. Let N satisfy

$$P_n < N \leqslant \sum_1^n P_i.$$

Then $M(N)$ is equal to $M(P_n)$ plus the number of non-representable integers in the interval (P_n, N). But any integer K in (P_n, N) which is representable must have P_n in its representation (noting

$$\sum_1^{n-1} P_i < P_n),$$

and since $K = P_n + (K - P_n)$, we see $K - P_n$ must also be representable. Conversely if $K - P_n$ (which is less than P_n) is representable in terms of $P_1, \cdots, P_{n-1}$, then K is clearly representable. Thus the number of non-representable integers in (P_n, N) is equal to the number of non-representable integers less than $N - P_n$, or $M(N - P_n)$. Hence

$$M(N) = M(P_n) + M(N - P_n),$$

establishing (i).

For N satisfying

$$\sum_1^n P_i < N < P_{n+1},$$

it is obvious that N is not representable. Moreover

$$M\left(\sum_1^n P_i + 1\right) = M\left(\sum_1^n P_i\right), \qquad M\left(\sum_1^n P_i + 2\right) = M\left(\sum_1^n P_i\right) + 1$$

(assuming the arguments of the left-hand terms are $< P_{n+1}$) and in general (adding 1 to $M(N)$ each time N is increased by 1),

$$M(N) = M\left[\sum_1^n P_i + \left(N - \sum_1^n P_i\right)\right] = M\left(\sum_1^n P_i\right) + \left(N - \sum_1^n P_i\right) - 1,$$

which is the first form of (ii). From Theorem 3 of [1],

$$M\left(\sum_1^n P_i\right) = \sum_1^n M(P_i) = \sum_1^n \left(P_i - 2^{i-1}\right) = \sum_1^n P_i - \sum_1^n 2^{i-1} = \sum_1^n P_i - (2^n - 1).$$

Then

$$M(N) = M\left(\sum_1^n P_i\right) + \left(N - \sum_1^n P_i\right) - 1 = \sum_1^n P_i - (2^n - 1) + \left(N - \sum_1^n P_i\right) - 1 = N - 2^n$$

as asserted.

Corollary 1. (Cf. [1], Theorem 4): If

$$N = \sum_1^n a_i P_i, \qquad \text{then} \qquad M(N) = \sum_1^n a_i M(P_i) = N - \sum_1^n a_i 2^{i-1}.$$

Proof. Let

$$N = \sum_1^n a_i P_i$$

with $a_n = 1$. Then

$$P_n \leqslant N \leqslant \sum_1^n P_i,$$

so that by (i) of Theorem 1, we have

$$M(N) = M(P_n) + M(N - P_n) = M(P_n) + M\left(\sum_1^K a_i P_i\right),$$

where $a_K = 1$ and $K < n$ (note K is simply the largest value of i less than n for which $a_i \neq 0$). Since

$$P_K \leqslant \sum_1^K a_i P_i \leqslant \sum_1^K P_i,$$

result (i) may be applied again and it is clear that successive iteration leads to

$$M(N) = \sum_1^n a_i M(P_i).$$

Using $M(P_i) = P_i - 2^{i-1}$, we have equivalently

$$M(N) = \sum_1^n a_i (P_i - 2^{i-1}) = \sum_1^n a_i 2^{i-1} = N - \sum_1^n a_i 2^{i-1}$$

as required.

Corollary 2. (Cf. [1], Theorem 3):

$$M\left(\sum_1^n P_i\right) = \sum_1^n M(P_i).$$

Proof. Immediate from Corollary 1 on taking all $a_i = 1$ for $i = 1, \cdots, n$.

3. EXAMPLE

Let $P_1 = 1, P_2 = 10, P_3 = 12, P_4 = 30, P_5 = 75, \cdots$ be the first 5 terms of a sequence which satisfies

$$P_{n+1} \geqslant 1 + \sum_{1}^{n} P_i$$

for all $n \geqslant 1$. Then, by direct enumeration

$M(1) = 0 = 1 - 2^0$	$M(13) = 8$
$M(2) = 0$	$M(14) = 8$
$M(3) = 1$	$M(15) = 9$
$M(4) = 2$	$M(16) = 10$
$M(5) = 3$	$M(17) = 11$
$M(6) = 4$	$M(18) = 12$
$M(7) = 5$	$M(19) = 13$
$M(8) = 6$	$M(20) = 14$
$M(9) = 7$	$M(21) = 15$
$M(10) = 8 = 10 - 2^1$	$M(22) = 16$
$M(11) = 8$	$M(23) = 16$
$M(12) = 8 = 12 - 2^2$	$M(24) = 16$
	$M(25) = 17$
	$M(26) = 18$
	$M(27) = 19$
	$M(28) = 20$
	$M(29) = 21$
	$M(30) = 22 = 30 - 2^3$

Now, note that all the values in the right-hand column may be calculated from those in the left-hand column; that is, if $12 < N < 30$, then we may apply Theorem 1 to see that

$$12 < N \leqslant 1 + 10 + 12 = 23 \to M(N) = M(12) + M(N-12)$$
$$23 < N < 30 \quad \to M(N) = N - 2^3$$

Thus, for example, $N = 21$ is not representable but $M(21) = M(12) + M(9) = 8 + 7 = 15$, where we have assumed the values $M(1)$ through $M(12)$ are known. Similarly $N = 27$ is not representable but > 23, so $M(27) = 27 - 2^3 = 19$. Then, knowing $M(1)$ through $M(30)$, we may use Theorem 1 again to calculate $M(31)$ through $M(74)$. Note that for case (i) of Theorem 1, only one addition is needed, since $N - P_n$ always $< P_n$ in this case, while for case (ii), the result for $M(N)$ is explicitly given by $N - 2^n$.

4. CONCLUSION

A recursive scheme has been derived for calculating $M(N)$, the number of integers less than N not representable as a sum of distinct terms from a fixed u-sequence $\left\{P_i\right\}_1^\infty$. This approach has the advantage of not requiring any prior information concerning which positive integers are representable; however, if a representation for N is known, the result of Hoggatt and Peterson provides an explicit formula for $M(N)$, while in at least some of the remaining cases [(ii) of Theorem 1] an explicit formula is obtained from Theorem 1 of this paper. Other values of $M(N)$ for non-representable N are easily calculated via the recursion relation (i) of Theorem 1. In addition, Theorem 1 provides alternative somewhat simpler deviations of Theorems 3 and 4 in [1].

REFERENCES

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