A NOTE ON WEIGHTED SEQUENCES

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1. It is well known that the Catalan number

$$a(n) = \frac{1}{n+1}\binom{2n}{n} \tag{1.1}$$

satisfies the recurrence

$$a(n+1) = \sum_{j=0}^{n} a(j)a(n-j) \qquad (n = 0, 1, 2, \cdots). \tag{1.2}$$

Conversely if (1.2) is taken as definition together with the initial condition $a(0) = 1$ then one can prove (1.1). Thus (1.1) and (1.2) are equivalent definitions.

This suggests as possible q-analogs the following two definitions:

$$\bar{a}(n,q) = \frac{1}{[n+1]}\begin{bmatrix} 2n \\ n \end{bmatrix}, \tag{1.3}$$

where

$$[n+1] = \frac{1-q^{n+1}}{1-q}, \qquad \begin{bmatrix} 2n \\ n \end{bmatrix} = \frac{(1-q^{2n})(1-q^{2n-1})\cdots(1-q^{n+1})}{(1-q)(1-q^2)\cdots(1-q^n)};$$

$$a(n+1,q) = \sum_{j=0}^{n} q^j a(j,q)a(n-j,q), \qquad a(0,q) = 1. \tag{1.4}$$

However (1.3) and (1.4) are not equivalent. Indeed

$$a(1,q) = 1, \qquad a(2,q) = 1+q, \qquad a(3,q) = (1+q)+q+q^2(1+q) = 1+2q+q^2+q^3,$$

$$\begin{aligned} a(4,q) &= (1+2q+q^2+q^3)+q(1+q)+q^2(1+q)+q^3(1+2q+q^2+q^3) \\ &= 1+3q+3q^2+3q^3+2q^4+q^5+q^6. \end{aligned}$$

On the other hand,

$$\bar{a}(1,q) = \frac{1}{[2]}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{1+q}\,\frac{1-q^2}{1-q} = 1,$$

$$\bar{a}(2,q) = \frac{1}{[3]}\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \frac{1}{1+q+q^2}\,\frac{(1-q^4)(1-q^3)}{(1-q)(1-q^2)} = 1+q^2,$$

$$\begin{aligned} \bar{a}(3,q) = \frac{1}{[4]}\begin{bmatrix} 6 \\ 3 \end{bmatrix} &= \frac{1}{1+q+q^2+q^3}\,\frac{(1-q^6)(1-q^5)(1-q^4)}{(1-q)(1-q^2)(1-q^3)} \\ &= 1+q^2+q^3+q^4+q^6, \end{aligned}$$

$$\bar{a}(4,q) = \frac{1}{[5]}\begin{bmatrix} 8 \\ 4 \end{bmatrix} = 1+q^2+q^3+2q^4+q^5+2q^6+q^7+2q^8+q^9+q^{10}+q^{12}.$$

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Another well known definition of the Catalan number is the following. Let $f(n,k)$ denote the number of sequences of positive integers $(a_1, a_2, \cdots, a_n)$ such that

(1.5) $$1 \leqslant a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n = k$$

and

(1.6) $$a_i \leqslant i \qquad (1 \leqslant i \leqslant n).$$

Then (see for example [1])

$$f(n,k) = \frac{n-k+1}{n}\binom{n+k-2}{n-1} \qquad (1 \leqslant k \leqslant n)$$

and in particular

$$f(n, n-1) = f(n,n) = \frac{1}{n}\binom{2n-2}{n-1} = a(n-1).$$

Next define $f(n, k, q)$ by means of [1]

(1.7) $$f(n,k,q) = \sum q^{a_1+a_2+\cdots+a_n} ,$$

where the summation is over all sequences $(a_1, a_2, \cdots, a_n)$ satisfying (1.5) and (1.6). It follows from this that the sum

(1.8) $$f(n,q) = \sum_{k=1}^{n} f(n,k,q)$$

satisfies

(1.9) $$f(n,q) = q^{-n} f(n+1,n,q) = q^{-n-1} f(n+1,n+1,q)$$

Moreover if we put

(1.10) $$f(n+1,k+1,q) = q^{(k+1)(n+1)-\frac{1}{2}k(k+1)} b(n,k,q^{-1})$$

then $b(n, k, q)$ satisfies

(1.11) $$b(n,k,q) = q^{n-k} b(n,k-1,q) + b(n-1,k,q).$$

We shall show that

(1.12) $$b(n,n,q) = a(n,q).$$

2. Returning to (1.4) we put

(2.1) $$A(x,q) = \sum_{n=0}^{\infty} a(n,q)x^n .$$

Then

$$A(x,q) = 1 + x\sum_{n=0}^{\infty} x^n \sum_{j=0}^{n} q^j a(j,q)a(n-j,q)$$

$$= 1 + x\sum_{j=0}^{\infty} a(j,q)q^j x^j \sum_{n=0}^{\infty} a(n,q)x^n ,$$

so that

(2.2) $$A(x,q) = 1 + xA(x,q)A(qx,q).$$

This gives

$$A(x,q) = \frac{1}{1 - xA(qx,q)},$$

which leads to the continued fraction

(2.3) $$A(x,q) = \frac{1}{1-}\,\frac{x}{1-}\,\frac{qx}{1-}\,\frac{q^2x}{1-}\cdots.$$

By a known result (see for example [3, p. 293])

$$\frac{1}{1-}\,\frac{x}{1-}\,\frac{qx}{1-}\,\frac{q^2x}{1-}\cdots = \frac{\Phi(qx,q)}{\Phi(x,q)},$$

where

(2.4) $$\Phi(x,q) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{n(n-1)}x^n}{(q)_n}$$

and

$$(q)_n = (1-q)(1-q^2)\cdots(1-q^n).$$

Therefore we get the identity

(2.5) $$A(x,q) = \frac{\Phi(qx,q)}{\Phi(x,q)}.$$

On the other hand it is proved in [1, (7.10)] that

(2.6) $$\sum_{k=0}^{\infty} b(n+k-1,k,q)x^k = \frac{\Phi(q^nx,q)}{\Phi(x,q)} \qquad (n > 0).$$

In particular, for $n = 1$, Eq. (2.6) reduces to

(2.7) $$\sum_{k=0}^{\infty} b(k,k,q)x^k = \frac{\Phi(qx,q)}{\Phi(x,q)}.$$

Comparing (2.7) with (2.5), we get

(2.8) $$b(k,k,q) = a(k,q).$$

3. For $x = -q$, Eq. (2.3) becomes

(3.1) $$A(-q,q) = \frac{1}{1+}\,\frac{q}{1+}\,\frac{q^2}{1+}\,\frac{q^3}{1+}\cdots.$$

It is known [3, p. 293] that the continued fraction

$$\frac{1}{1+}\,\frac{q}{1+}\,\frac{q^2}{1+}\cdots = \prod_{n=0}^{\infty} \frac{(1-q^{5n+2})(1-q^{5n+3})}{(1-q^{5n+1})(1-q^{5n+4})}.$$

Thus (2.5) yields the identity

(3.2) $$\sum_{n=0}^{\infty} (-1)^n a(n,q)q^n = \prod_{n=0}^{\infty} \frac{(1-q^{5n+2})(1-q^{5n+3})}{(1-q^{5n+1})(1-q^{5n+4})}.$$

Another connection in which $a(n,q)$ occurs is the following. It can be shown that $a(n+1,q)$ is the number of *weighted* triangular arrays

(3.3) $$\begin{matrix} a_{11} & & a_{12} & \cdots & a_{1n} \\ & a_{21} & \cdots & a_{2,n-1} & \\ & & \cdots & & \\ & & a_{n1} & & \end{matrix},$$

where $a_{ij} = 0$ or 1 and

(3.4) $$a_{ij} \geqslant a_{i+1,j-1}, \qquad a_{ij} \geqslant a_{i+1,j}.$$

More precisely

$$a(n+1, q) = \sum q^{\Sigma a_{ij}}, \tag{3.5}$$

where the outer summation is over all (0.1) arrays (3.3) satisfying (3.4) and the sum Σa_{ij} is simply the number of ones in the array.

For example, for $n = 2$, we have the arrays

$$\begin{array}{c}0\ \ 0\\ 0\end{array} \qquad \begin{array}{c}1\ \ 0\\ 0\end{array} \qquad \begin{array}{c}0\ \ 1\\ 0\end{array} \qquad \begin{array}{c}1\ \ 1\\ 0\end{array} \qquad \begin{array}{c}1\ \ 1\\ 1\end{array}$$

This gives

$$1 + 2q + q^2 + q^3 = a(3, q).$$

For $n = 3$ we have

$$\begin{array}{c}0\,0\,0\\ 0\,0\\ 0\end{array} \;\Big|\; \begin{array}{c}1\,0\,0\\ 0\,0\\ 0\end{array} \quad \begin{array}{c}0\,1\,0\\ 0\,0\\ 0\end{array} \quad \begin{array}{c}0\,0\,1\\ 0\,0\\ 0\end{array} \;\Big|\; \begin{array}{c}1\,1\,0\\ 0\,0\\ 0\end{array} \quad \begin{array}{c}1\,0\,1\\ 0\,0\\ 0\end{array} \quad \begin{array}{c}0\,1\,1\\ 0\,0\\ 0\end{array} \;\Big|\; \begin{array}{c}1\,1\,0\\ 1\,0\\ 0\end{array} \quad \begin{array}{c}0\,1\,1\\ 0\,1\\ 0\end{array} \quad \begin{array}{c}1\,1\,1\\ 0\,0\\ 0\end{array} \;\Big|$$

$$\begin{array}{c}1\,1\,1\\ 1\,0\\ 0\end{array} \quad \begin{array}{c}1\,1\,1\\ 0\,1\\ 0\end{array} \;\Big|\; \begin{array}{c}1\,1\,1\\ 1\,1\\ 0\end{array} \;\Big|\; \begin{array}{c}1\,1\,1\\ 1\,1\\ 1\end{array}$$

This gives

$$1 + 3q + 2q^2 + 3q^3 + 2q^6 + q^5 + q^6 = a(4, q).$$

Let $T_k(n)$ denote the number of solutions in non-negative integers a_{ij} of the equation

$$n = \sum_{i=1}^{k} \sum_{j=1}^{k-i+1} a_{ij},$$

where the a_{ij} satisfy the inequalities

$$a_{ij} \geqslant a_{i+1,j}, \qquad a_{ij} \geqslant a_{i+1,j-1}.$$

It has been proved in [2] that

$$\sum_{n=1}^{\infty} T_k(n)x^n = \frac{1}{(1-x^{2k-1})(1-x^{2k-3})^2 \cdots (1-x^5)^{k-2}(1-x^3)^{k-1}(1-x)^k} \tag{3.6}$$

REFERENCES

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3. G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers,* Clarendon Press, Oxford, 1938.

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