NON-HYPOTENUSE NUMBERS

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The non-hypotenuse numbers $n = 1, 2, 3, 4, 6, 7, 8, 9, 11, 12, 14, 16, 18, \cdots$ are those natural numbers for which there is *no* solution of

(1) $$n^2 = u^2 + v^2 \qquad (u > v > 0).$$

Although they occur very frequently for small n they nonetheless have zero density—almost all natural numbers n do have solutions. Only 1/15.547 of the numbers around 10^{100} are NH numbers, and, around $2^{19937} - 1$, only 1/120.806.

In a review of a table by A. H. Beiler [1], I had occasion to remark that if $NH(x)$ is the number of such $n \leqslant x$ then

(2) $$NH(x) \sim Ax/\sqrt{\log x}$$

for some coefficient A. Recently, T. H. Southard wished to know this A because of an investigation [2] originating in a study of Jacobi theta functions. Inasmuch as most of the analysis and arithmetic has already been done in [3], one can be more precise and easily compute accurate values of A and C in the asymptotic expansion:

(3) $$NH(x) = \frac{Ax}{\sqrt{\log x}}\left[1 + \frac{C}{\log x} + O\left(\frac{1}{\log^2 x}\right)\right].$$

Landau's function $B(x)$ is the number of $n \leqslant x$ for which there *is* a solution of

(4) $$n = u^2 + v^2.$$

Note: n to the *first* power, and all u, v allowed. Then

(5) $$B(x) = \frac{bx}{\sqrt{\log x}}\left[1 + \frac{c}{\log x} + O\left(\frac{1}{\log^2 x}\right)\right]$$

and I evaluated

(6) $$b = 0.764223654, \qquad c = 0.581948659$$

in [3]. The n of (4) are those n divisible only by 2, by primes $p \equiv 1 \pmod 4$, and by even powers of primes $q \equiv 3 \pmod 4$. If $b_m = 1$ for any m = any such n, and $b_m = 0$ otherwise, one has the generating function

(7) $$\sum_{m=1}^{\infty} \frac{b_m}{m^s} = f(s) = \frac{1}{1-2^{-s}} \prod_p \frac{1}{1-p^{-s}} \prod_q \frac{1}{1-q^{-2s}}.$$

In contrast, the NH numbers are those divisible by no prime p, and so they are generated by

(8) $$g(s) = \frac{1}{1-2^{-s}} \prod_q \frac{1}{1-q^{-s}}.$$

Since

(9) $$L(s) = 1 - 3^{-s} + 5^{-s} - 7^{-s} + \cdots = \prod_p \frac{1}{1-p^{-s}} \prod_q \frac{1}{1+q^{-s}}$$

we can write

(10) $$g(s) = f(s)/L(s).$$

Landau [4] showed that $f(s)$ has a branch point at $s = 1$ and a convergent series

(11) $$f(s) = \frac{ais^2}{\sqrt{1-s}}\,[1 + a_1(1-s)/a + \cdots]$$

in its neighborhood for computable coefficients $a, a_1, \cdots$. In terms of these, one evaluates the coefficients of (5) as

(12) $$b = \frac{a\Gamma(\tfrac{1}{2})}{\pi}, \qquad c = (a_1 - a)/2a$$

with the usual method using Cauchy's theorem and integration around the branch point. But $L(s)$ is analytic at $s = 1$ and so we have, at once,

(13) $$d = \frac{a}{L(1)}, \qquad \frac{d_1}{d} = \frac{a_1}{a} + \frac{L'(1)}{L(1)}$$

for the new generator

(14) $$g(s) = \frac{dis^2}{\sqrt{1-s}}\,[1 + d_1(1-s)/d + \cdots].$$

Therefore

(15) $$A = b/L(1), \qquad C = c + L'(1)/2L(1)$$

give the wanted coefficients of (3). Of course, $L(1) = \pi/4$, and in [3] one has

(16) $$L'(1)/L(1) = \log\left[\left(\frac{\pi}{\tilde{\omega}}\right)^2 \frac{e^{\gamma}}{2}\right]$$

in terms of the Euler constant γ and the lemniscate constant $\tilde{\omega}$. So, from [3] one has

(17) $$A = \frac{2\sqrt{2}}{\pi}\prod_q (1-q^{-2})^{-1/2} = 0.97303978$$

and

(18) $$C = \frac{1}{2}\left[1 + \log\left(\frac{\pi}{\tilde{\omega}}\right) - \frac{1}{2}\frac{d}{ds}\log\prod_q \frac{1}{1-q^{-2s}}\Bigg|_{s=1}\right] = 0.70475345.$$

In [2] Southard gives

$$NH(99999) - NH(99000) = 295,$$

while (3), (17) and (18) give

$$NH(99999) - NH(99000) = 289.36.$$

It is known that the third-order term in (3) is positive but it was not computed.

REFERENCES

1. A. H. Beiler, "Consecutive Hypotenuses of Pythagorean Triangles," UMT 74, *Math. Comp.,* Vol. 22, 1968, pp. 690–692.
2. Thomas H. Southard, *Addition Chains for the First n Squares,* Center Numerical Analysis, CNA-84, Austin, Texas, 1974.
3. Daniel Shanks, "The Second-Order Term in the Asymptotic Expansion of *B(x),*" *Math. Comp.,* Vol. 18, 1964, pp. 75–86.
4. Edmund Landau, "Uber die Einteilung, usw.," *Archiv der Math. and Physik* (3), Vol. 13, 1908, pp. 305–312.

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$$(1/-1) = 1,$$
$$(-1/1) = 1,$$
$$(1/1) = 1.$$

The second entry of the Extended Jacobi Symbol is multiplicative by definition; it will be proved in the corollaries that both entries are also periodic.

The following results are easily derived:

Explicitly,

$$(0/1) = 1,$$
$$(0/b) = 0 \text{ if } b \neq 1,$$
$$(0/-b) = 0 \text{ if } -b \neq 1,$$
$$(2/\pm b) = (-1)^{(b^2-1)/8},$$
$$(-2/b) = (-1)^{(b^2+4b-5)/8},$$
$$(-2/-b) = (-1)^{(b^2-4b-5)/8}.$$

If $a \neq 0$, then

$$(-a^2/-1) = -1,$$
$$(-1/-b^2) = -1;$$
$$(-a/1) = 1,$$
$$(a/-1) = (a/-1) \text{ (see below)},$$
$$(-a/-1) = -(a/-1);$$

$$(1/b) = 1,$$
$$(-1/b) = (-1)^{(b-1)/2},$$
$$(1/-b) = 1,$$
$$(-1/-b) = (-1)^{(b+1)/2}.$$

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