GENERAL IDENTITIES FOR FIBONACCI AND LUCAS NUMBERS WITH POLYNOMIAL SUBSCRIPTS IN SEVERAL VARIABLES

RICHARD R. STONE
Student, Montana State University, Bozeman, Montana 59715

Among the well known Fibonacci identities we have

$$F_{m+n} \equiv F_{m+1}F_n + F_mF_{n-1}$$

which may be written as

$$F_{m+1}F_n - F_1F_{m+n} \equiv F_mF_{n-1}\ .$$

In this form, we see a property which is common among Fibonacci and Lucas identities. Namely, that the sum of the subscripts of the first product $F_{m+1}F_n$ is identically equal to the sum of the subscripts of the second product F_1F_{m+n}.

What general identities do we have with this property? How does this property relate to the reducibility of a given form?

It is with these questions that we are principally concerned.

Definition 1. For every i, $1 \leq i \leq m$, let the domain of n_i be the set of integers. Then we let

$$P = \{\text{ polynomials in } n_1, n_2, \cdots, n_m \text{ with integral coefficients }\}.$$

For convenience in deriving general Fibonacci and Lucas identities for the forms

$$F_fF_g \pm F_hF_k, \qquad L_fL_g \pm L_hL_k, \qquad F_fL_g \pm F_hL_k\,,$$

where $f, g, h, k \in P$, with the property that $f+g \equiv h+k$, we first express h and k in terms of f and g.

Lemma 1. If $f, g, h, k \in P$ such that $f+g \equiv h+k$, then there exists $f_1, f_2, g_1, g_2 \in P$ such that

$$f_1+f_2 \equiv f, \qquad g_1+g_2 \equiv g, \qquad f_1+g_1 \equiv h, \quad \text{and} \quad f_2+g_2 \equiv k.$$

Proof. Let

$$f_1 \equiv h, \qquad f_2 \equiv f-h, \qquad g_1 \equiv 0, \qquad g_2 \equiv g,$$

clearly,

$$f_1, f_2, g_1, g_2 \in P \qquad \text{and} \qquad f_1+f_2 \equiv f, \qquad g_1+g_2 \equiv g, \qquad f_1+g_1 \equiv h\,.$$

$$f_2+g_2 \equiv f-h+g$$

but, by hypothesis,

$$f+g \equiv h+k \Rightarrow f-h+g \equiv k \Rightarrow f_2+g_2 \equiv k. \qquad \text{q.e.d.}$$

Theorem 1. Let $f, g, h, k \in P$ such that $f+g \equiv h+k$, then

$$F_fF_g - F_hF_k \equiv (-1)^{g+1}F_{f-h}F_{f-k}\ .$$

Proof. By hypothesis,

$$f+g \equiv h+k \qquad \text{and} \qquad f, g, h, k \in P$$

Hence, by Lemma 1, there exist $f_1, f_2, g_1, g_2 \in P$ such that

$$f_1+f_2 \equiv f, \qquad g_1+g_2 \equiv g, \qquad f_1+g_1 \equiv h, \qquad f_2+g_2 \equiv k\,.$$

Then, clearly,

$$F_fF_g - F_hF_k \equiv F_{f_1+f_2}F_{g_1+g_2} - F_{f_1+g_1}F_{f_2+g_2}\ .$$

Using the Binet definition

$$\left(F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \text{ where } n \in [\text{Integers}], \quad \alpha = \frac{1+\sqrt{5}}{2}, \quad \beta = \frac{1-\sqrt{5}}{2} \right)$$

we have

$$\begin{aligned}
F_{f_1+f_2}F_{g_1+g_2} - F_{f_1+g_1}F_{f_2+g_2} &\equiv \left(\frac{\alpha^{f_1+f_2} - \beta^{f_1+f_2}}{\alpha-\beta}\right)\left(\frac{\alpha^{g_1+g_2} - \beta^{g_1+g_2}}{\alpha-\beta}\right) \\
&\quad - \left(\frac{\alpha^{f_1+g_1} - \beta^{f_1+g_1}}{\alpha-\beta}\right)\left(\frac{\alpha^{f_2+g_2} - \beta^{f_2+g_2}}{\alpha-\beta}\right) \\
&\equiv \frac{(\alpha^{f_1+f_2+g_1+g_2} - \beta^{f_1+f_2}\alpha^{g_1+g_2} - \alpha^{f_1+f_2}\beta^{g_1+g_2} + \beta^{f_1+f_2+g_1+g_2})}{(\alpha-\beta)^2} \\
&\quad - \frac{(\alpha^{f_1+f_2+g_1+g_2} - \beta^{f_1+g_1}\alpha^{f_2+g_2} - \alpha^{f_1+g_1}\beta^{f_2+g_2} + \beta^{f_1+g_1+f_2+g_2})}{(\alpha-\beta)^2} \\
&\equiv \frac{(-\beta^{f_1+f_2}\alpha^{g_1+g_2} + \alpha^{f_1+g_1}\beta^{f_2+g_2} - \alpha^{f_1+f_2}\beta^{g_1+g_2} + \beta^{f_1+g_1}\alpha^{f_2+g_2})}{(\alpha-\beta)^2} \\
&\equiv \frac{\beta^{f_2}\alpha^{g_1}(-\beta^{f_1}\alpha^{g_2} + \alpha^{f_1}\beta^{g_2}) + \alpha^{f_2}\beta^{g_1}(-\alpha^{f_1}\beta^{g_2} + \beta^{f_1}\alpha^{g_2})}{(\alpha-\beta)^2} \\
&\equiv \frac{(-\beta^{f_1}\alpha^{g_2} + \alpha^{f_1}\beta^{g_2})(\beta^{f_2}\alpha^{g_1} - \alpha^{f_2}\beta^{g_1})}{(\alpha-\beta)^2} \\
&\equiv \frac{(\alpha\beta)^{g_2}(-\beta^{f_1-g_2} + \alpha^{f_1-g_2})(\beta\alpha)^{g_1}(\beta^{f_2-g_1} - \alpha^{f_2-g_1})}{(\alpha-\beta)^2} \\
&\equiv \frac{(\alpha\beta)^{g_1+g_2+1}(\alpha^{f_1-g_2} - \beta^{f_1-g_2})(\alpha^{f_2-g_1} - \beta^{f_2-g_1})}{(\alpha-\beta)^2} \\
&\equiv (-1)^{g_1+g_2+1}F_{f_1-g_2}F_{f_2-g_1} .
\end{aligned}$$

But

$g_1 + g_2 \equiv g$ and $f_1 - g_2 \equiv (f_1+f_2) - (f_2+g_2) \equiv f-k$ and $f_2 - f_1 \equiv (f_1 + f_2) - (f_1+g_1) \equiv f-h$.

Thus, by substituting

$$(-1)^{g_1+g_2+1}F_{f_1-g_2}F_{f_2-g_1} \equiv (-1)^{g+1}F_{f-k}F_{f-h} \equiv (-1)^{g+1}F_{f-h}F_{f-k} \quad \text{q.e.d.}$$

Theorem 2. Let $f, g, h, k \in P$ such that $f+g \equiv h+k$, then

(a)

$$L_fL_g - L_hL_k \equiv 5(-1)^gF_{f-h}F_{f-k}$$

and

(b)

$$F_fL_g - F_hL_k \equiv (-1)^{g+1}F_{f-h}L_{f-k} .$$

Proof. The proof of 2(a) and 2(b) is virtually the same as that of Theorem 1 (where $L_n = \alpha^n + \beta^n$).

Corollary 1. Let $f, g, h, k \in P$ such that $f+g \equiv h+k$. Then

$$F_fF_g - F_hF_k \equiv -\frac{(L_fL_g - L_hL_k)}{5} .$$

Proof. Compare Theorems 2(a) and 1.

EXAMPLES AND APPLICATIONS

The degree of freedom offered by Theorems 1 and 2 together with the identity given in their hypothesis is large indeed. We will endeavor, with some examples, to indicate that degree of freedom.

EXAMPLE 1. By [1, p. 7], a general Turàn operator is defined by

$$Tf = T_xf(x) = f(x+u)f(x+v) - f(x)f(x+u+v).$$

"For the Fibonacci numbers it is a classic formula first discovered apparently by Tagiuri (Cf. Dickson [4, p. 404]) and later given as a problem in the *American Mathematical Monthly* (Problem 1396) that

$$T_nF_n = F_{n+u}F_{n+v} - F_nF_{n+u+v} = (-1)^nF_uF_v."$$

This is immediate from Theorem 1.

Let $f \equiv n+u,\ g \equiv n+v,\ h \equiv n$ and $k \equiv n+u+v$. Clearly,

$$f, g, h, k \in P \qquad \text{and} \qquad f+g \equiv h+k.$$

Thus, applying Theorem 1, we have

$$F_{n+u}F_{n+v} - F_nF_{n+u+v} \equiv (-1)^{n+v+1}F_{(n+u)-n}F_{(n+u)-(n+u+v)} \equiv (-1)^{n+v+1}F_uF_{-v}.$$

Now using the well known identity $(-1)^{m+1}F_m \equiv F_{-m}$ yields

$$(-1)^{n+v+1}F_uF_{-v} \equiv (-1)^{n+v+1}(-1)^{-v+1}F_uF_v \equiv (-1)^nF_uF_v,$$

the desired result.

EXAMPLE 2. By Theorem 2(a),

$$L_fL_g - L_hL_k \equiv (-1)^g5F_{f-h}F_{f-k}$$

if $f, g, h, k \in P$ and $f+g \equiv h+k$. Then too, $f-k \equiv h-g$ and $f-h \equiv k-g$.

Substituting, we obtain

$$L_fL_g - L_hL_k \equiv (-1)^g5F_{k-g}F_{h-g},$$

a trivial but equivalent form of Theorem 2(a).

Another equivalent form of Theorem 2(a) is

$$L_fL_g - 5F_hF_k \equiv (-1)^gL_{k-g}L_{h-g}.$$

To obtain this equivalent form, we write

$$f + (-g) \equiv (h-g) + (k-g).$$

Clearly,

$$f, (-g), (h-g), (k-g) \in P;$$

hence, Theorem 2(a) may be applied to these new polynomials, yielding,

$$L_fL_{(-g)} - L_{(h-g)}L_{(k-g)} \equiv (-1)^{-g}F_{(h-g)-(-\beta)}F_{(k-g)-(-g)}$$

then,

$$(-1)^gL_fL_g - L_{(h-g)}L_{(k-g)} \equiv (-1)^g5F_hF_k \Rightarrow L_fL_g - 5F_hF_k \equiv (-1)^gL_{k-g}L_{h-g}.$$

Similarly, Theorems 1 and 2 may be put into several other equivalent forms.

It would be natural to ask what $F_fF_g + F_hF_k$ would yield, subject to the condition

$$f, g, h, k \in P \qquad \text{and} \qquad f+g \equiv h+k,$$

with a proof analagous to that of Theorem 1. The result is, in at least one form,

$$F_fF_g + F_hF_k \equiv \frac{L_0L_{f+g}}{5} + (-1)^{g+1}\frac{L_{(f-h)}L_{(f-k)}}{5}.$$

However, this form is easily derived with the following method.

EXAMPLE 3.

$$f+g \equiv h+k \Rightarrow (0) + (f+g) \equiv h+k,$$

by Theorem 2(a),

$$\frac{L_{f+g}L_0}{5} - \frac{L_hL_k}{5} \equiv F_hF_k.$$

Now we use Theorem 2(a) to find an expression for F_fF_g and obtain

$$F_fF_g - \frac{L_hL_k}{5} \equiv \frac{(-1)^{g+1}L_{f-h}L_{f+k}}{5} .$$

Adding these identities produces

$$F_fF_g + F_hF_k = \frac{L_0L_{f+g}}{5} + (-1)^{g+1}\,\frac{L_{f-h}L_{f-k}}{5} .$$

Similarly, we find sums $L_fL_g + L_hL_k$ by using Theorem 2(b). Also, other sums with various equivalent forms may be found.

APPLICATION TO FIBONACCI AND LUCAS TRIPLES

Application of Theorems 1 and 2 to the Fibonacci and Lucas triples [2], generated by R. T. Hansen, allow Theorems 1 and 2 to be written in equivalent summation form for fixed integers.

Theorem 3. Let A, B be fixed integers; then

$$F_AF_B \equiv \sum_{K=0}^{B-1} (-1)^{B+1-K}F_{A-B+2K+1}$$

$$F_AL_B \equiv \sum_{K=0}^{A-1} (-1)^{B+K}L_{A-B-2K+1}$$

$$L_AL_B \equiv \sum_{K=0}^{A} (-1)^{B+K}L_{A-B-2(K+1)} + \sum_{K=0}^{A-2} (-1)^{B+K}L_{A-B-2K} \cdot$$

Proof. See [2] and directly apply Theorems 1 and 2.

Clearly, from these forms, the summation equivalents of Theorems 1 and 2, for fixed integer A, B, C, D such that $A + B = C + D$, may be obtained as immediate corollaries. We do not list these identities.

FURTHER APPLICATION OF THEOREMS 1 AND 2

We now apply Theorems 1 and 2 to find simple subscript properties between identically equal Fibonacci and Lucas products.

Lemma 2. Let $f, g \in P$ such that $f \neq 2$ and $g \neq 2$. If $F_f \equiv F_g$, then $|f| \equiv |g|$.

Proof.

$$F_f \equiv F_g \Rightarrow |F_f| \equiv |F_g| \Rightarrow F_{|f|} \equiv F_{|g|} .$$

Clearly,

$$\{F_N\}_{N=0}^{\infty}, \qquad N \neq 2, \qquad N \in [\text{Integers}],$$

is a strictly increasing sequence. Then $F_{|f|} \equiv F_{|g|}$ and $|f| \neq |g|$ is a contradiction to the fact that $\{F_N\}_{N=0}^{\infty}$, $N \neq 2$, is strictly increasing. Thus,

$$F_f \equiv F_g \Rightarrow F_{|f|} \equiv F_{|g|} \Rightarrow |f| \equiv |g| . \qquad \text{Q.E.D.}$$

Theorem 4. Given $f, g, h, k \in P$. If $F_fF_g \equiv F_hF_k$, then $|f| \equiv |h|$ and $|g| \equiv |k|$, or $|f| \equiv |g|$ and $|g| \equiv |k|$ whenever

$$|f|, |g|, |h|, |k|, \notin \{0,2\}.$$

Proof. If $F_fF_g \equiv F_hF_k$, then

(1) $$|F_fF_g| \equiv |F_hF_k| \Rightarrow F_{|f|}F_{|g|} \equiv F_{|h|}F_{|k|}.$$

Since $f, g, h, k \in P$, they are functions of $n_1, n_2, \cdots,$ and n_m. Let n_i' for $1 \leqslant i \leqslant m$ be an arbitrary set of fixed values of n_i for $1 \leqslant i \leqslant m$, respectively. Then f_1, g_1, h_1, k_1 are the corresponding fixed integers. Assume W.L.O.G. that

$$|f_1| + |g_1| \geqslant |h_1| + |k_1|$$

and that $\{n_i'\}$ is such that

$$|f_1|, |g_1|, |h_1|, |k_1|$$

are not 2 or 0. Clearly, there exist K such that $K > 0$, $K \in$ [Integers] and

(2) $$|f_1| + |g_1| = |h_1| + |k_1| + K.$$

By Theorem 1,

$$F_{|f_1|}F_{|g_1|} - F_{|h_1|}F_{|k_1|+K} = (-1)^{|g_1|+1}F_{|f_1|-|h_1|}F_{|f_1|-(|k_1|+K)} = 0$$

if and only if

$$|f_1| - |h_1| = 0 \qquad \text{or} \qquad |f_1| - (|k_1| + K) = 0.$$

Without loss of generality, assume that

$$|f_1| - |h_1| = 0 \Rightarrow |f_1| = |h_1|.$$

Then by (2), $|g_1| = |k_1| + K$.

Suppose $K \neq 0$, then

$$F_{|f_1|}F_{|g_1|} = F_{|h_1|}F_{|k_1|+K} \neq F_{|h_1|}F_{|k_1|}$$

by Lemma 2.

Thus, if

$$F_{|f_1|}F_{|g_1|} = F_{|h_1|}F_{|k_1|}$$

it is required that $K = 0$. Thus,

$$|f_1| = |h_1| \qquad \text{and} \qquad |g_1| = |k_1|.$$

Further, since the selection of n_i' was arbitrary with the conditions of the theorems hypothesis, its conclusion holds. Q.E.D.

Note that the condition

$$|f|, |g|, |h|, |k| \notin \{2\}$$

is not really any restriction, practically speaking. That is $F_2 = F_1$, so if one agrees always to write F_2 as F_1 we could require only that $|f|, |g|, |h|, |k| \notin \{0\}$ in the hypothesis of Theorem 4.

Lemma 3. Let $f, g \in P$, if $L_f \equiv L_g$, then $|f| \equiv |g|$.

Proof. Construct an argument similar to Lemma 2.

Theorem 5. Let $f, g, h, k \in P$. If $L_fL_g \equiv L_hL_k$, then $|f| \equiv |h|$ and $|g| \equiv |k|$, or $|f| \equiv |k|$ and $|g| \equiv |h|$.

Proof. Construct a proof analogous to Theorem 4 by using Lemma 3 and Theorem 2(a).

Theorem 6. Given $f, g, h, k \in P$. If $F_fL_g \equiv F_hL_k$, then $|f| \equiv |h|$ and $|g| \equiv |k|$, whenever $|f|, |h| \notin \{0,2\}$.

Proof. Construct a proof analogous to Theorem 4 by using Theorem 2(b). Informally speaking, Theorems 1, 2, 4, 5 and 6 seem to suggest that an algebraic structure for Fibonacci identities, based on the subscripts, can be formed. If the reader is interested in investigating this, he will be more successful in using the following form of Theorem 1:

$$F_{f_1+f_2}F_{g_1+g_2} - F_{f_1+g_1}F_{f_2+g_2} \equiv (-1)^{g_1+g_2+1}F_{f_1-g_2}F_{f_2-g_1},$$

where

$$f, g, h, k, f_1, f_2, g_1, g_2 \in P$$

and

$$f_1 + f_2 = f, \qquad g_1 + g_2 = g, \qquad f_1 + g_1 = h, \qquad f_2 + g_2 = k$$

and

$$f + g \equiv h + k.$$

Further, note that if we let

$$Q = \left\{ F_R F_S \mid R, S \in P \quad \text{and} \quad R + S \equiv f + g \right\}$$

then clearly

$$F_{f_1+f_2} F_{g_1+g_2}, \; F_{f_1+g_1} F_{f_2+g_2} \in Q.$$

Also,

$$F_{f_1+f_2} F_{g_1+g_2} \equiv (-1)^{g_1+g_2+1} F_{f_1+f_2} F_{-g_1-g_2} \Rightarrow (-1)^{g_1+g_2+1} F_{f_1+f_2} F_{-g_1} F_{-g_2} \in Q$$

and then

$$(-1)^{g_1+g_2+1} F_{f_1-g_2} F_{f_2-g_1} \in Q.$$

The reader may enjoy investigating further in this or other directions.

SOME ADDITIONAL IDENTITIES

Theorem 7. Let $f, g, h \in P$ such that $f \equiv g + h$. Then,

(a) $$F_f - F_g L_h \equiv (-1)^g F_{h-g}$$

(b) $$L_f - L_g L_h \equiv (-1)^{g+1} L_{h-g}$$

(c) $$\frac{L_f}{5} - F_g F_h \equiv \frac{(-1)^g L_{h-g}}{5}.$$

Proof. By using the Binet definition we have

$$F_f - F_g L_h \equiv \frac{\alpha^f - \beta^f}{\alpha - \beta} - \frac{\alpha^g - \beta^g}{\alpha - \beta} \cdot \frac{\alpha^h + \beta^h}{1} \equiv \frac{(\alpha^f - \beta^f) - (\alpha^{g+h} - \beta^g \alpha^h + \alpha^g \beta^h - \beta^{g+h})}{\alpha - \beta}.$$

By hypothesis $f \equiv g + h$, hence by substituting $g + h$ for f in the above expression and simplifying we have

$$F_f - F_g L_h \equiv \frac{\beta^g \alpha^h - \alpha^g \beta^h}{\alpha - \beta}$$

$$\equiv (\alpha\beta)^g \frac{(\alpha^{h-g} - \beta^{h-g})}{\alpha - \beta} \equiv (-1)^g F_{h-g}.$$

The proofs of (b) and (c) are similar. Q.E.D.

Although not included, theorems corresponding to those in this paper may be developed for Fibonacci and Lucas triples as well. (The author did develop the $F_g F_h L_k - F_l F_m F_n$ form.) Clearly, the proofs for these, which are virtually the same as for Theorems 1 and 2, soon become cumbersome. We leave it to the reader to develop these to suit his needs.

REFERENCES

1. H. W. Gould, "Generating Functions for Products of Powers of Fibonacci Numbers," *The Fibonacci Quarterly*, Vol. 1, No. 1 (Feb. 1963), p. 8.
2. R. T. Hansen, "Generating Identities for Fibonacci and Lucas Triples," *The Fibonacci Quarterly*, Vol. 10, No. 5 (Dec. 1972), pp. 571–578.

****