Hence $g(r) = v$. Suppose $g^k(r) = v$ whenever $vc^k \leqslant r < (v+1)c^k$. Suppose, in addition, that $vc^{k+1} \leqslant r_0 < (v+1)c^{k+1}$. Then

$$g^{k+1}(r_0) = g^k\left(\left[\frac{r_0}{c}\right]\right) \qquad \text{and} \qquad vc^k \leqslant \frac{r_0}{c} < (v+1)c^k\ .$$

It follows that

$$vc^k \leqslant \frac{r_0}{c} < (v+1)c^k\ .$$

Hence by the induction hypothesis

$$g^{k+1}(r_0) = g^k \cdot g(r_0) = g^k\left(\left[\frac{r_0}{c}\right]\right) = v\ .$$

To prove Theorem 1, employ Theorem 2 to obtain positive integers n and m such that

$$v < \frac{f^n(u)}{c^m} < v+1$$

and apply Lemma 4.

REFERENCE

1. Ivan Niven, "Irrational Numbers," *The Carus Mathematical Monographs,* No. 11, published by The Mathematical Association of America.

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We can add any quantity B to each term:

$$x(a+B)^m + y(b+B)^m + (x+y-2)(ax+by+B)^m = (x+y-2)B^m + y(ax+by+B-b)^m + x(ax+by+B-a)^m$$
(where $m = 1, 2$).

A special case of a Fibonacci-type series is

$$1^m \quad 2^m \quad 3^m \quad \cdots \quad n^m\ .$$

Consider the series when $m = 2$:

(1) $\qquad 1 \quad 4 \quad 9 \quad 16 \quad 25 \quad \cdots \quad \cdots$

where

$$F_n = 3(F_{n-1} - F_{n-2}) + F_{n-3}$$

[we obtain our coefficients from Pascal's Triangle], i.e.,

$$(x+3)^2 = 3[(x+2)^2 - (x+1)^2] + x^2\ .$$

I have found by conjecture that

$$1^m - 4^m - 4^m - 4^m + 9^m + 9^m + 9^m - 16^m = -0^m - 12^m - 12^m - 12^m + 7^m + 7^m + 7^m + 15^m$$
(where $m = 1, 2$).

[I hope the reader will accept the strange -0^m for the time being.]

If we express the series (1) above in the form

$$a \quad b \quad 3(c-b)+a \quad \text{etc.,}$$

our multigrade appears as follows

$$a^m - 3b^m + 3c^m - [3(c-b)+a]^m = -0^m - 3(3c-4b+a)^m + 3(2c-3b+a)^m + [3(c-b)]^m$$
(where $m = 1, 2$).

We could, of course, write the above as

$$(x^2)^m - 3[(x+1)^2]^m + 3[(x+2)^2]^m - [3[(x+2)^2 - (x+1)^2] + x^2]^m$$
$$= -0^m - 3[x^2 - 4(x+1)^2 + 3(x+2)^2]^m + 3[x^2 - 3(x+1)^2 - 4(x+2)^2]^m + [3[(x+2)^2 - (x+1)^2]^m$$
(where $m = 1, 2$).

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