ELEMENTARY PROBLEMS AND SOLUTIONS

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Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman; 709 Solano Dr., S.E.; Albuquerque, New Mexico 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within four months of the publication date.

DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \quad F_0 = 0, \quad F_1 = 1 \qquad \text{and} \qquad L_{n+2} = L_{n+1} + L_n, \quad L_0 = 2, \quad L_1 = 1.$$

Also a and b designate the roots $(1+\sqrt{5})/2$ and $(1-\sqrt{5})/2$, respectively, of $x^2 - x - 1 = 0$.

PROBLEMS PROPOSED IN THIS ISSUE

B-322 Proposed by Sidney Kravitz, Dover, New Jersey.

Solve the following alphametic in which no 6 appears:

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  A R K I N
  A L D E R
  S A L L E
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A L L A D I
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(All the names are taken from the front cover of the April 1975 *Fibonacci Quarterly*.)

B-323 Proposed by J. A. H. Hunter, Fun with Figures, Toronto, Ontario, Canada.

Prove that
$$F_{n+r}^2 - (-1)^r F_n^2 = F_r F_{2n+r}.$$

B-324 Proposed by Herta T. Freitag, Roanoke, Virginia.

Determine a constant k such that, for all positive integers n,

$$F_{3n+2} \equiv k^n F_{n-1} \pmod 5.$$

B-325 Proposed by Verner E. Hoggatt, Jr., San Jose State University, San Jose, California.

Let $a = (1+\sqrt{5})/2$ and $b = (1-\sqrt{5})/2$. Prove that there does not exist an *even* single-valued function G such that

$$x + G(x^2) = G(ax) + G(bx) \quad \text{on} \quad -a \leqslant x \leqslant a.$$

B-326 Based on the solution to B-303 by David Zeitlin, Minneapolis, Minnesota.

For positive integers n, let $\sigma(n)$ be the sum of the positive integral divisors of n. Prove that

$$\sigma(mn) \geqslant 2\sqrt{\sigma(m)\sigma(n)} \quad \text{for} \quad m > 1 \quad \text{and} \quad n > 1.$$

B-327 Proposed by George Berzsenyi, Lamar University, Beaumont, Texas.

Find all integral values of r and s for which the equality

$$\sum_{i=0}^{n} \binom{n}{i} (-1)^i L_{ri} = s^n L_n$$

holds for all positive integers n.

SOLUTIONS

A CORRECTION

Jeffrey Shallit points out that the second solution $\pi^2/2^2$ to problem B-274 is incorrect and that a correct solution using π, /, and 2 is $\pi^2/2^{2^2}$.

AN APPLICATION OF THE BINET FORMULAS

B-298 Proposed by Richard Blazej, Queens Village, New York.

Show that

$$5F_{2n+3}F_{2n-3} = L_{4n} + 18.$$

Solution by Gerald E. Bergum, South Dakota State University, Brookings, South Dakota.

Using

$$F_n = (a^n - b^n)/\sqrt{5}, \qquad L_n = a^n + b^n,$$

and the fact that

$$ab = -1, \quad 5F_{2n+3}F_{2n-3} = (a^{2n+3} - b^{2n+3})(a^{2n-3} - b^{2n-3}) = a^{4n} + b^{4n} - (a^6 + b^6)(ab)^{2n-3}$$
$$= L_{4n} - L_6(-1)^{2n-3} = L_{4n} + L_6 = L_{4n} + 18.$$

Also solved by George Berzsenyi, Wray G. Brady, Paul S. Bruckman, Warren Cheves, Herta T. Freitag, Ralph Garfield, Frank Higgins, Graham Lord, John W. Milsom, C. B. A. Peck, Jeffrey Shallit, A. C. Shannon, David Zeitlin, and the Proposer.

A CONVOLUTION FORMULA

B-299 Proposed by Verner E. Hoggatt, Jr., San Jose State University, San Jose, California.

Establish a simple closed form for

$$F_{2n+3} - \sum_{k=1}^{n} (n+2-k)F_{2k}.$$

Solution by Frank Higgins, Naperville, Illinois.

Using Problem B-295, we have

$$F_{2n+3} - \sum_{k=1}^{n} (n+2-k)F_{2k} = F_{2n+3} - \sum_{k=1}^{n} (n+1-k)F_{2k} - \sum_{k=1}^{n} F_{2k}$$
$$= F_{2n+3} - (F_{2n+2} - (n-1)) - (F_{2n+1} - 1) = n+2.$$

Also solved by Gerald E. Bergum, George Berzsenyi, Wray C. Brady, Paul S. Bruckman, Herta T. Freitag, Ralph Garfield, Peter A. Lindstrom, Graham Lord, C. B. A. Peck, Jeffrey Shallit, A. C. Shannon, and the Proposer.

ANOTHER CONVOLUTION

B-300 Proposed by Verner E. Hoggatt, Jr., San Jose State University, San Jose, California.

Establish a simple closed form for

$$L_{2n+2} - \sum_{k=1}^{n} (n+3-k)F_{2k}.$$

Solution by George Berzsenyi, Lamar University, Beaumont, Texas.

Utilizing well known formulae and the solution to Problem B-299, one finds that

$$
\begin{aligned}
L_{2n+2} - \sum_{k=1}^{n} (n+3-k)F_{2k} &= L_{2n+2} - \sum_{k=1}^{n} (n+2-k)F_{2k} - \sum_{k=1}^{n} F_{2k} \\
&= L_{2n+2} - (F_{2n+3} - n - 2) - (F_{2n+1} - 1 \\
&= L_{2n+2} + n + 3 - (F_{2n+1} + F_{2n+3}) \\
&= L_{2n+2} + n + 3 - L_{2n+2} = n + 3.
\end{aligned}
$$

Also solved by Gerald E. Bergum, Wray G. Brady, Paul S. Bruckman, Herta T. Freitag, Ralph Garfield, Frank Higgins, Peter A. Lindstrom, Graham Lord, C. B. A. Peck, Jeffrey Shallit, A. C. Shannon, and the Proposer.

GREATEST INTEGER IDENTITY

B-301 Proposed by Phil Mana, Albuquerque, New Mexico.

Let $[x]$ denote the greatest integer in x, i.e., the integer m with $m \leqslant x < m+1$. Also let

$$A(n) = (n^2 + 6n + 12)/12 \quad \text{and} \quad B(n) = (n^2 + 7n + 12)/6.$$

Does $[A(n)] + [A(n+1)] = [B(n)]$ for all integers n? Explain.

Solution by Graham Lord, Secane, Pennsylvania.

The identity is correct, as can be seen upon placing $n = 6m, 6m+1, \cdots, 6m+5$ successively. For example, with $n = 6m+4$:

$$
\begin{aligned}
[A(n)] + [A(n+1)] &= [A(6m+4)] + [A(6m+5)] = (3m^2 + 7m + 4) + (3m^2 + 8m + 5) \\
&= [6m^2 + 15m + (56/6)] = B(6m+4) = B(n).
\end{aligned}
$$

Also solved by Paul S. Bruckman, David Zeitlin, and the Proposer.

COMPOSITE FIBONACCI NEIGHBORS

B-302 Proposed by Verner E. Hoggatt, Jr., San Jose State University, San Jose, California.

Prove that $F_n - 1$ is a composite integer for $n \geqslant 7$ and that $F_n + 1$ is composite for $n \geqslant 4$.

Solution by John Ivie, Student, University of California, Berkeley, California.

Using the Binet Formulas, the following identities can be established:

$$
\begin{aligned}
F_{4k} - 1 &= L_{2k-1}F_{2k+1}; & F_{4k} + 1 &= L_{2k+1}F_{2k-1}. \\
F_{4k+1} - 1 &= L_{2k+1}F_{2k}; & F_{4k+1} + 1 &= L_{2k}F_{2k+1}. \\
F_{4k+2} - 1 &= L_{2k+2}F_{2k}; & F_{4k+2} + 1 &= L_{2k}F_{2k+2}. \\
F_{4k+3} - 1 &= L_{2k+1}F_{2k+2}; & F_{4k+3} + 1 &= L_{2k+2}F_{2k+1}.
\end{aligned}
$$

Since $F_n > 1$ for $n > 2$ and $L_n > 1$ for $n > 1$, one thus sees that $F_n - 1$ is composite for $n \geqslant 7$ and $F_n + 1$ is composite for $n \geqslant 4$.

Also solved by Gerald E. Bergum, George Berzsenyi, Paul S. Bruckman, Graham Lord, A. C. Shannon, David Zeitlin, and the Proposer.

A SIGMA FUNCTION INEQUALITY

B-303 Proposed by David Singmaster, Polytechnic of the South Bank, London, England.

In B-260, it was shown that $\sigma(mn) > \sigma(m) + \sigma(n)$, where $\sigma(n)$ is the sum of the positive integral divisors of n. What relation holds between $\sigma(mn)$ and $\sigma(m)\sigma(n)$?

Solution by Frank Higgins, Naperville, Illinois.

Let

$$m = \prod_{i=1}^{k} p_i^{\alpha_i} \qquad \text{and} \qquad n = \prod_{i=1}^{k} p_i^{\beta_i},$$

where each α_i and β_i is non-negative and where $p_1, p_2, \cdots, p_k$ are distinct prime numbers. Since

$$\frac{p_i^{\alpha_i+\beta_i+1} - 1}{p_i - 1} \leqslant \frac{p_i^{\alpha_i+1} - 1}{p_i - 1} \cdot \frac{p_i^{\beta_i+1} - 1}{p_i - 1},$$

where equality holds iff $\alpha_i\beta_i = 0$, it follows that $\sigma(mn) \leqslant \sigma(m)\sigma(n)$, where equality holds iff $(m,n) = 1$.

Also solved by Paul S. Bruckman, Graham Lord, C. B. A. Peck, David Zeitlin, and the Proposer.

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