SOME OPERATIONAL FORMULAS

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1. INTRODUCTION

In this paper we consider some simple variations of the derivative and the difference operator; deriving formulas for powers and factorials.

Let \( s(n,k) \) denote the Stirling number of the first kind and \( S(n,k) \) denote the Stirling number of the second kind. They are defined by:

\[
(x)_n = \sum_{k=1}^{n} s(n,k)x^k
\]

\[
x^n = \sum_{k=1}^{n} S(n,k)(x)_k
\]

where

\[
(x)_n = x(x-1)(x-2) \cdots (x-n+1).
\]

Substituting (1.1) in (1.2) or (1.2) in (1.1) shows that

\[
a_n = \sum s(n,k)b_k \quad \text{and} \quad b_n = \sum S(n,k)a_k
\]

are equivalent (inverse) relations.

Define

\[
A_n(x) = \sum_{k=1}^{n} s(n,k)x^k
\]

\[
A^{(n)}(x) = \sum_{k=1}^{n} (-1)^{n-k}s(n,k)x^k
\]

\[
B_n(x) = \sum_{k=1}^{n} S(n,k)x^k
\]

\[
B^{(n)}(x) = \sum_{k=1}^{n} (-1)^{n-k}S(n,k)x^k
\]

Then \( A_n(x) = (x)_n \), the falling factorial; \( A^{(n)}(x) = x^{(n)} \), the rising factorial and \( B_n(x) \) is the single variable Bell polynomial [3, p. 35]. We have \( A_n(B(x)) = x^n = B_n(A(x)) \), etc., where \( (B(x))^k = B_k(x) \), \( (A(x))^k = A_k(x) \).

We will employ the following special notation:

\[
[\theta \phi]^n = \theta^n \phi^n
\]

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and if 

\[ f_n(x) = \sum_{i=0}^{n} a_i x^i \]

then 

\[ f_n(\theta \phi) = \sum_{i=0}^{n} a_i (\theta \phi)^i = \sum_{i=0}^{n} a_i \theta^i \phi^i . \]

REMARK. When \( \theta \) and \( \phi \) commute or \( n = 1 \) then 

\[ [\theta \phi]^n = (\theta \phi)^n \quad \text{and} \quad f_n(\theta \phi) = f_n(\theta) f_n(\phi) . \]

2. THE OPERATORS \( xD, Dx, x\Delta, \Delta x \)

Operators of the form \( (xD)^n, D^n x^n, (\Delta x)^n \), etc., are often difficult to work with and we seek equivalent forms. First we note that 

\[ (xD)_n = A_n(xD) = \sum_{k=1}^{n} S(n,k)(xD)^k = x^n D^n \]

follows by induction from 

\[ (xD)_{k+1} = (xD)_k (xD - k) = x^k D^k (xD - k) = x^k (D^k x) D - kx^k D^k \]

\[ = x^k (x D^k + k D^{k-1}) D - kx^k D^k = x^{k+1} D^{k+1} . \]

But (2.1) admits the inverse 

\[ (xD)^n = \sum S(n,k)x^k D^k = B_n [xD] . \]

Equation (2.2) can also be shown directly using the recurrence for \( S(n,k) \) [4, p. 218]. Similarly, 

\[ (x\Delta)_n = A_n(x\Delta) = \sum_{k=0}^{n} a(n,k)(x\Delta)^k = x^{(n)} \Delta^n \]

follows by induction from 

\[ (x\Delta)_{k+1} = (x\Delta - k)(x\Delta)_k = (x\Delta - k)x^{(k)} \Delta^k = \left\{ x \Delta x^{(k)} - k x^{(k)} \right\} \Delta^k \]

\[ = \left\{ x x^{(k)} \Delta + k x(x+1)^{(k-1)} \Delta - k x^{(k)} \right\} \Delta^k \]

\[ = \left\{ x x^{(k)} \Delta + k x(x+1)^{(k-1)} \Delta^k \right\} \Delta^k = (x + k)x^{(k)} \Delta k = x^{(k+1)} \Delta^{k+1} . \]

But (2.3) admits the inverse 

\[ (x\Delta)^n = \sum S(n,k)x^{(k)} \Delta^k = B_n [x\Delta] \]

where \( x^j = x^{(j)} \).

Since 

\[ (Dx)^n = x^{-1} (xD)^{n+1} D^{-1} \quad \text{and} \quad (\Delta x)^n = x^{-1} (x\Delta)^{n+1} \Delta^{-1} \]

we have from (2.2) and (2.4), respectively, 

\[ (Dx)^n = x^{-1} B_{n+1} [xD] D^{-1} = \sum_{k=1}^{n+1} S(n+1, k)x^{k-1} D^{k-1} . \]
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\( (\Delta x)^n = x^{-1}B_{n+1}[x\Delta] \Delta^{-1} = \sum_{k=1}^{n+1} S(n+1, k)(x+1)^{(k-1)} \Delta^{k-1} . \)

Using Leibnitz's formula for the derivative of a product we get; cf. [1, p.]

\[
D^n x^n = \sum_{k=0}^{n} \binom{n}{k} (D^k x^n)D^{n-k} = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} D^{n-k} = \sum_{k=0}^{n} \binom{n}{k} \frac{n!}{(n-k)!} x^{n-k} D^{k-n}.
\]

Replacing \( n - k \) by \( k \) we have

\(\frac{n!}{k!} = \sum_{k=0}^{n} \frac{1}{k!} x^k D^k.\)

Using

\[
D^{k+1} x^{k+1} = D^k \left\{ x^{k+1}D + (k+1)x^k \right\} = D^k x^k \left\{ xD + k + 1 \right\}
\]
we have by induction

\( D^n x^n = (xD + 1)^{(n)} = (Dx)^{(n)} = A^{(n)}(Dx). \)

Since

\( (xD)^{(n)} = (xD)(xD+1)^{(n-1)} = (xD)(Dx)^{(n-1)} = xD^n x^{n-1} \)

we have

\( (xD)^{(n)} = xD^n x^{n-1} . \)

Using the difference analogue of Leibnitz's formula [2, p. 96] we get cf. [1, p. 4],

\[
\Delta^n x^{(n)} = \sum_{k=0}^{n} \binom{n}{k} \Delta^{n-k} E^{(n)} x^{(n)} \Delta^k = \sum_{k=0}^{n} \binom{n}{k} \Delta^{n-k} (x + k)^{(n)} \Delta^k.
\]

Hence

\(\Delta^n x^{(n)} = \sum_{k=0}^{n} \binom{n}{k} \frac{n!}{k!} (x + n)^{(k)} \Delta^k.\)

Using

\[
\Delta^{k+1}(x)_{k+1} = \Delta^k (x + 1)_{k+1} = \Delta^k \left\{ x + 1 \right\} \Delta + (k+1)(x)_{k+1} \Delta \}
\]
we have by induction

\(\Delta^n x^{(n)} = A^{(n)}(\Delta x). \)

But

\(\Delta^n x^{(n)} = \Delta^n (x + n - 1)^{(n)} = (\Delta (x + n - 1))^{(n)} .\)

hence \( \Delta x = x \Delta + \Delta + 1 \) we have

\(\Delta^n x^{(n)} = ((x + n) \Delta + 1)^{(n)} = ((x + n) \Delta + n)^{(n)}. \)

Taking the inverse of (2.8) we have

\(\frac{(Dx)^{n}}{n!} = \sum_{k=1}^{n} (-1)^{n-k} S(n,k) D^k x^k = B^{(n)}(Dx). \)
Taking the inverse of (2.10) we have

\[(\Delta x)^n = \sum_{k=1}^{n} (-1)^{n-k} S(n,k) \Delta^k (x) = B^{(n)}(\Delta x),\]

where \(x^I = (x)_I\).

Since

\[(xD)^{m+n} = (xD)^m (xD)^n \quad \text{and} \quad \{(xD)^m\}^n = (xD)^{mn}\]

we have by (2.2)

\[(2.14) \quad B_{m+n}(xD) = B_m(xD)B_n(xD), \quad (B_m(xD))^n = B_{mn}(xD).\]

Similarly (2.4) gives

\[(2.15) \quad B_{m+n}(x\Delta) = B_m(x\Delta)B_n(x\Delta), \quad \{B_m(x\Delta)\}^n = B_{mn}(x\Delta).\]

Similar results also hold for \(B^{(k)}[Dx]\) and \(B^{(k)}[\Delta x]\).

3. THE OPERATORS \(x(l+D), x(l+\Delta), (l+D)x, (l+\Delta)x\)

Analogous to (2.1) is

\[(3.1) \quad (x(l+D))_n = A_n(x(l+D)) = x^n(l+D)^n = [x(l+D)]^n\]

which follows by induction from

\[(x(l+D))_{k+1} = (x(l+D))_k (x(l+D) - k) = x^k(l+D)^k(x(l+D) - k)\]

\[= x^k \{x(l+D)^{k+1} + k(l+D)^k - k(l+D)^k\} = x^{k+1}(l+D)^{k+1} .\]

But (3.1) admits the inverse

\[(3.2) \quad (x(l+D))^n = \sum_{k=1}^{n} S(n,k) x^k (l+D)^k = B_n(x(l+D)) .\]

Since

\[(l(l+D))^{n+1} = (l+D)^n(l+D) = (l+D)^n x + n + 1 = (l+D)^n x^n(l+D)x + n\]

we have by induction

\[(3.3) \quad (l(l+D)x)^n = \sum_{k=1}^{n+1} S(n+1,k)x^{k-1}(l+D)^{k-1} .\]

Using

\[(l+D)^{n+1} x^{n+1} = (l+D)^n(l+D)x = (l+D)^n x^n x + xD + n + 1 = (l+D)^n x^n(l+D)x + n\]

we have by induction

\[(3.4) \quad (l+D)^{n} x^n = (l+D)_x A^{(n)}((l+D)x)\]

which admits the inverse

\[(3.5) \quad (l(l+D)x)^n = \sum_{k=1}^{n} (-1)^{n-k} S(n,k)(l+D)^k x^k = B^{(n)}((l+D)x) .\]

By (3.4) and since \((l+D)x = (x +xD) + 1,\)

\[(x(l+D))^{(n)} = (x +xD)^{(n)} = x(l+D)((l+D)x)^{n-1} = x(l+D)((l+D))^{n-1} x^{n-1}.\]

Hence
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(3.6) \( (x(l + D))^n = x(l + D)^n x^{n-1} \).

By (3.1) and since

\[ (x + Dx)_n = (x + Dx)(x + xD)_n \]

we have

(3.7) \( (l + D)x)_n = (l + D)x^n(l + D)^{n-1} \).

Using (1.4)

(3.8) \( (x(l + \Delta))^n = \sum_{k=1}^{n} (-1)^{n-k} s(n,k)(x(l + \Delta))^k = A^{(n)}(x(l + \Delta)) \).

But,

(3.9) \( (x(l + \Delta))^n = x^{(n)}(l + \Delta)^n \)

follows by induction from

\[ (x(l + \Delta))^{k+1} = (x(l + \Delta))(x(l + \Delta))^k = x(l + \Delta) x^{(k)}(l + \Delta)^k \]

\[ = x \left\{ x^{(k)} + x^{(k)} \Delta + k(x + 1) \Delta^{k-1} + k(x + 1) \Delta^{k-1} \right\} (l + \Delta)^k \]

\[ = x \left\{ x^{(k)} + k(x + 1) \Delta^{k-1} \right\} (l + \Delta)^{k+1} = x^{(k+1)}(l + \Delta)^{k+1} \].

Hence

(3.10) \( (x(l + \Delta))^n = \sum_{k=1}^{n} (-1)^{n-k} s(n,k)(x(l + \Delta))^k = A^{(n)}(x(l + \Delta)) \),

where \( x^k = x^{(k)} \).

Relation (3.8) admits the inverse

(3.11) \( (x(l + \Delta))^n = \sum_{k=1}^{n} (-1)^{n-k} s(n,k)(x(l + \Delta))^k = B^{(n)}(x(l + \Delta)) \),

where \( (x(l + \Delta))^k = (x(l + \Delta))^{(k)} \).

Using (3.9), (3.11) may be rewritten

(3.12) \( (x)^n(l + \Delta)^n = \sum_{k=1}^{n} (-1)^{n-k} s(n,k)(x(l + \Delta))^{(k)} \).

Using (1.1)

(3.13) \( (x(l + \Delta))_n = \sum_{k=1}^{n} s(n,k)(x(l + \Delta))^k = A_n(x(l + \Delta)) \)

and using (3.9)

(3.14) \( (x(l + \Delta))_n = \sum_{k=1}^{n} s(n,k)x^{(k)}(l + \Delta)^k = A_n(x(l + \Delta)) \),

where the inverses of (3.13 and (3.14 are, respectively,

(3.15) \( (x(l + \Delta))^n = \sum_{k=1}^{n} S(n,k)(x(l + \Delta))^k = B_n(x(l + \Delta)) \).
and

(3.16) \[ x^{(n)}(l + \Delta)^n = \sum_{k=1}^{n} S(n,k)(x(l + \Delta))_k = B_n(x(l + \Delta)). \]

Iterating \((l + \Delta)x = x + x\Delta + \Delta + l = (x + 1)(l + \Delta)\) \(n\) times we have

(3.17) \[ (l + \Delta)^nx = (x + n)(l + \Delta)^n. \]

More generally,

(3.18) \[ (l + \Delta)^nx^{(n)} = (x + n)^{(n)}(l + \Delta)^n \]

as the following induction step shows:

\[ (l + \Delta)^{n+1}x^{(n+1)} = (l + \Delta)^n(l + \Delta)x^{(n+1)} = (l + \Delta)^n(x + 1)^{(n)}(x + n + 1)(l + \Delta) \]

\[ = (x + 1 + n)^{(n)}(l + \Delta)^n(x + n + 1)(l + \Delta). \]

Using (3.17) we get

\[ (x + 1 + n)^{(n)}(x + n + 1 + n)(l + \Delta)^n(l + \Delta) = (x + n + 1)^{(n+1)}(l + \Delta)^{n+1}. \]

Replacing \(x\) by \(x + 1\) in (3.9) and using (3.17) for \(n = 1\) we have

(3.19) \[ (l + \Delta)x^{(n)} = (x + 1)^{(n)}(l + \Delta)^n = (l + \Delta)^n(x)_n. \]

Similarly (3.10) becomes

(3.20) \[ ((l + \Delta)x)^{(n)} = A^{(n)}[(x + 1)(l + \Delta)] = A^{(n)}[(l + \Delta)x], \]

where \((x + 1)^k \equiv (x + 1)^{(k)}\).

Equation (3.11) becomes

(3.21) \[ ((l + \Delta)x)^{(n)} = B^{(n)}[(x + 1)(l + \Delta)] = B^{(n)}[(l + \Delta)x]. \]

Equation (3.14) becomes

(3.22) \[ ((l + \Delta)x)_n = A_n[(l + \Delta)x]. \]

4. THE OPERATORS \(xD^2x, Dx^2D, x\delta^2x - 1, \delta(x - 1)^{[2]}\delta\)

We first note that \(xD\) and \(Dx\) commute, i.e.,

(4.1) \[ xD^2x = xD^2x = x^{2}D + 2xD = Dx^2D = DxD^2 \]

and we restrict our attention to \(xD^2x\).

Since \(xD^2x = xD^2x = xD(1 + xD) = B_1[xD](1 + B_1[xD]),\)

\[ (xD^2x)_n = \left\{ B_1[xD](1 + B_1[xD]) \right\}^n \]

By (2.14) this gives

(4.2) \[ (xD^2x)_n = B_n[xD](1 + B_1[xD])^n \]

or alternatively

(4.3) \[ (xD^2x)_n = \sum_{k=0}^{n} \binom{n}{k} B_{n+k}[xD]. \]

This becomes
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\[ (xD^2 x)^n = \sum_{k=0}^{n} \binom{n}{k} \sum_{j=0}^{n+k} S(n+k,j) x^j D^j \]

or utilizing (2.2),

\[ (xD^2 x)^n = \sum_{k=0}^{n} \binom{n}{k} (xD)^{n+k} \cdot \]

Since \( xD \) and \( Dx \) commute with each other,

\[ (xD^2 x)^n = (xD Dx)^n = (xD)^n (Dx)^n = [(xD)(Dx)]^n \cdot \]

Using (2.2) and (2.12) this gives

\[ (xD^2 x)^n = B_n [xD] B^{(n)} [Dx] \]

Comparison with (4.2) yields

\[ B^{(n)} [Dx] = \sum_{k=0}^{n} \binom{n}{k} B_k [xD] \cdot \]

Since by (2.1) and (2.8),

\[ x^n D^{2n} x^n = x^n D^n x^n = (xD)_n (Dx)^n \cdot \]

and since

\[ (xD - k)(Dx + k) = (xD - k)(xD + 1 + k) = xD^2 x - k^2 \cdot \]

we have, analogous to (2.1) and (2.8),

\[ x^n D^{2n} x^n = \prod_{k=0}^{n} (xD^2 x - k^2) \cdot \]

Remark.

\[ D^n x^{2n} x^n = x^n D^{2n} x^n \cdot \]

We note that \( xA \) and \( A (x - 1) \) commute, i.e.,

\[ xA^2 (x - 1) = xA (1 + xA) = (1 + xA)x = (x - 1)^{[2]} A \cdot \]

Writing

\[ xA^2 (x - 1) = xA (1 + xA) = B_1 [xA] (1 + B_1 [xA]) \]

we have using (2.14)

\[ (xA^2 (x - 1))^n = B_n [xA] (1 + B_1 [xA])^n \]

or

\[ (xA^2 (x - 1))^n = \sum_{k=0}^{n} \binom{n}{k} B_{n+k} [xA] = \sum_{k=0}^{n} \binom{n}{k} \sum_{j=0}^{n+k} S(n+k,j) x^j D^j \]

or using (2.4)

\[ (xA^2 (x - 1))^n = \sum_{k=0}^{n} \binom{n}{k} (xA)^{n+k} \cdot \]

Since by (2.3) and (2.10)
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\[ x^{(n)} \Delta^n \Delta^n (x - 1)_n = (x\Delta)_n (\Delta (x - 1))^{(n)} = (x\Delta)_n (x\Delta + 1)^{(n)} \]

and since

\[ (x\Delta - k)(x\Delta + 1 + k) = (x\Delta^2 (x - 1) - k^{(2)}) \]

we have, analogous to (4.8),

\[ (4.13) \]

\[ x^{(n)} \Delta^{2n} (x - 1)_n = \prod_{k=0}^{n} (x\Delta^2 (x - 1) - k^{(2)}) \]

5. THE OPERATORS \( x(l + D)x, x(l + \Delta)^l(x - 1) \)

The operators \( x(l + D) \) and \( (l + D)x \) commute, i.e.,

\[ (5.1) \]

\[ x(l + D)^2 x = (l + D)x^2(l + D), \]

and we have using (3.2)

\[ (5.2) \]

\[ (x(l + D)^2 x)^n = \sum_{k=0}^{n} \binom{n}{k} B_{n+k} [x(l + D)] = \sum_{k=0}^{n} \binom{n}{k} (x(l + D))^{n+k} \]

and

\[ (5.3) \]

\[ (x(l + D)^2 x)^n = \sum_{n=0}^{n} \binom{n}{k} \sum_{j=0}^{n+k} S(n + k, j) x^j (l + D)^j. \]

The operators \( x(l + \Delta) \) and \( (l + \Delta)(x - 1) \) commute, i.e.,

\[ (5.4) \]

\[ x(l + \Delta)^2 (x - 1) = (l + \Delta)(x - 1)^{(2)}(l + \Delta). \]

Using (3.18),

\[ (5.5) \]

\[ x(l + \Delta)^2 (x - 1) = x(l + \Delta) x(l + \Delta) = (x(l + \Delta))^2. \]

Hence by (3.9)

\[ (5.6) \]

\[ x(l + \Delta)^2 (x - 1)^n = (x(l + \Delta))^{2n} = x^{(2n)}(l + \Delta)^{2n}. \]

Since

\[ x^{(n)}(l + \Delta)^n(l + \Delta)^n (x - 1)_n = x^{(n)}(l + \Delta)^n(l + \Delta)^n (x - n)^{(n)} \]

\[ = x^{(n)}(l + \Delta)^n x^{(n)}(l + \Delta)^n = x^{(n)}(x + n)^{(n)}(l + \Delta)^n \]

we have

\[ (5.7) \]

\[ x^{(n)}(l + \Delta)^{2n}(x - 1)_n = x^{(2n)}(l + \Delta)^{2n} \]

and comparing with (5.6)

\[ (5.8) \]

\[ x(l + \Delta)^2 (x - 1)^n = x^{(n)}(l + \Delta)^{2n}(x - 1)_n. \]

REFERENCES