ON ISOMORPHISMS BETWEEN THE NATURALS AND THE INTEGERS

SAMUEL T. STERN
State University College at Buffalo, New York 14222

The mapping

$$g(m) = \left[\frac{m}{2} \right] (-1)^m ,$$

where $[x]$ denotes the greatest integer in x, from the set of naturals N onto the set of all integers I is one-to-one. This mapping fails to preserve natural order and the operations of ordinary addition and multiplication. For while $2 < 3$, $g(2) \nless g(3)$; also $g(2+3) \neq g(2) + g(3)$ and $g(2 \cdot 3) \neq g(2)g(3)$. However, it is possible to define an appropriate order relation $\wr$ and binary operations $(+)$ and $(\cdot)$ on I, while retaining natural order and ordinary addition and multiplication on N such that g will become an isomorphism of N to I, preserving order, addition, and multiplication as follows:

$$x \wr y \text{ means } \begin{cases} |x| > |y| & \text{if } |x| \neq |y| \\ x < 0 \text{ and } y > 0 & \text{if } |x| = |y| \end{cases} \tag{1}$$

$$x\,(+)\,y = \left[\frac{1 + |2x - \tfrac{1}{2}| + |2y - \tfrac{1}{2}|}{2} \right] (-1)^{1+|2x-\frac{1}{2}|+|2y-\frac{1}{2}|} \tag{2}$$

$$x\,(\cdot)\,y = \left[\frac{1 + |4x-1| + |4y-1| + |4x-1||4y-1|}{8} \right] (-1)^{\frac{1+|4x-1|+|4y-1|+|4x-1||4y-1|}{4}} . \tag{3}$$

Noting that $[m/2]$ is equal to $m/2$ if m is even and $(m-1)/2$ if m is odd, it is easy to show that $m > n$ if and only if $g(m) \wr g(n)$. Furthermore,

$$g(m+n) = g(m)\,(+)\,g(n) \qquad \text{and} \qquad g(mn) = g(m)\,(\cdot)\,g(n).$$

An analogous treatment can be given the integers interpreted as equivalence classes of nonnegative integers. We let A be the set of all ordered pairs (a,b) of nonnegative integers and let $(a,b) \sim (c,d)$ if and only if $a + d = b + c$. This defines an equivalence relation $\sim$ on A. Let B be the set of all equivalence classes of A with respect to this relation. Consider the mapping

$$f(m) = K \left(\frac{m}{4}\left(1 + (-1)^m\right), \frac{m-1}{4}\left(1 + (-1)^{m-1}\right) \right), \tag{4}$$

where $K(a,b)$ denotes the equivalence class of A which contains (a,b). f is one-to-one from N onto B. For let $K(a,b)$ represent an arbitrary element of B. If $a = b$ then $f(1) = K(a,b)$. If $a = b + k$, k a positive integer, then, $f(2k) = K(a,b)$. If $b = a + k$, k a positive integer, then $f(2k+1) = K(a,b)$. Furthermore if

$$K \left(\frac{m}{4}\left(1 + (-1)^m\right), \frac{m-1}{4}\left(1 + (-1)^{m-1}\right) \right) = K \left(\frac{n}{4}\left(1 + (-1)^n\right), \frac{n-1}{4}\left(1 + (-1)^{n-1}\right) \right)$$

then $(-1)^m(2m-1) = (-1)^n(2n-1)$. Hence m and n must be either both even or both odd, and it follows that $m = n$.

The absolute value of an element $K(a,b)$ of B, denoted by $|a,b|$ is defined as follows:

$$|a,b| = \begin{cases} K(a,b) & \text{if } a > b \\ K(b,a) & \text{if } a \leqslant b \end{cases} . \tag{5}$$

The order relation Δ is defined on B as follows:

(6) $$K(a,b)\ \Delta\ K(c,d) \text{ if and only if } a+d > b+c.$$

The order relation ∇ is defined on B as follows:

(7) $$K(a,b)\ \nabla\ K(c,d) \text{ means } \begin{cases} |a,b|\ \Delta\ |c,d| & \text{if } |a,b| \neq |c,d| \\ a < b \text{ and } c > d & \text{if } |a,b| = |c,d|. \end{cases}$$

We show that with the relation of (7) on B and natural order on N the mapping (4) is an order isomorphism. For suppose that

$$K\left(\frac{m}{4}(1+(-1)^m), \frac{m-1}{4}(1+(-1)^{m-1})\right)\ \nabla\ K\left(\frac{n}{4}(1+(-1)^n), \frac{n-1}{4}(1+(-1)^{n-1})\right)$$

If these have the same absolute value, then by (7),

$$(-1)^{m+1}(2m-1) > 1 \quad \text{and} \quad (-1)^{n+1}(2n-1) < 1.$$

From the first of these inequalities we see that m is odd and since $2n-1$ is not zero $(-1)^{n+1}(2n-1)$ must be a negative integer, whence n is even. Thus

$$\left|0, \frac{m-1}{2}\right| = \left|\frac{n}{2}, 0\right|,$$

that is,

$$K\left(\frac{m-1}{2}, 0\right) = K\left(\frac{n}{2}, 0\right),$$

which implies $m > n$.

On the other hand if the two equivalence classes have different absolute values then

$$K\left(\frac{m}{2}, 0\right)\ \Delta\ K\left(\frac{n}{2}, 0\right)$$

if m and n are both even,

$$K\left(\frac{m-1}{2}, 0\right)\ \Delta\ K\left(\frac{n-1}{2}, 0\right)$$

if m and n are both odd, and

$$K\left(\frac{m-1}{2}, 0\right)\ \Delta\ K\left(\frac{n}{2}, 0\right)$$

if m is odd and n even. In each case we have $m > n$. If m is even and n odd then

$$K\left(\frac{m}{2}, 0\right)\ \Delta\ K\left(\frac{n-1}{2}, 0\right)$$

which implies $m \geqslant n$. But $m \neq n$. Hence $m > n$. Conversely, let $m > n$. Then if m and n are even,

$$\left|\frac{m}{2}, 0\right|\ \Delta\ \left|\frac{n}{2}, 0\right| \quad \text{and} \quad K\left(\frac{m}{2}, 0\right)\ \nabla\ K\left(\frac{n}{2}, 0\right).$$

If m and n are odd, then

$$\left|0, \frac{m-1}{2}\right|\ \Delta\ \left|0, \frac{n-1}{2}\right| \quad \text{and} \quad K\left(0, \frac{m-1}{2}\right)\ \nabla\ K\left(0, \frac{n-1}{2}\right).$$

If m is odd and n even and if also $m = n+1$, then

$$\left|0, \frac{m-1}{2}\right| = \left|\frac{n}{2}, 0\right|.$$

But if $m > n+1$, then

$$\left|0, \frac{m-1}{2}\right|\ \Delta\ \left|\frac{n}{2}, 0\right|.$$

Either way

$$K\left(0, \frac{m-1}{2}\right)\ \nabla\ K\left(\frac{n}{2}, 0\right).$$

If m is even and n odd, then

$$\left|\frac{m}{2}, 0\right| \vartriangle \left|0, \frac{n-1}{2}\right| \qquad \text{and} \qquad K\left(\frac{m}{2}, 0\right) \triangledown K\left(0, \frac{n-1}{2}\right).$$

Thus we have shown that $m > n$ if and only if $f(m) \triangledown f(n)$.

The operations $\oplus$, of addition and $\otimes$, of multiplication are defined on B as follows:

$$K(a,b) \oplus K(c,d) = \begin{cases} K(a+c,\ b+d) & \text{if } m,n \text{ are even} \\ K(b+d+1,\ a+c) & \text{if } m,n \text{ are odd} \\ K(b+c,\ a+d) & \text{if } m \text{ is even, } n \text{ odd} \\ K(a+d,\ b+c) & \text{if } m \text{ is odd, } n \text{ even} \end{cases} \tag{8}$$

$$K(a,b) \otimes K(c,d) = \begin{cases} K(2(a-b)(c-d), 0) & \text{if } m,n \text{ are even} \\ K(c, d+2(a-b)(c-d)+b-a) & \text{if } m,n \text{ odd} \\ K(a+2(a-b)(d-c), b) & \text{if } m \text{ is even, } n \text{ odd} \\ K(c+2(a-b)(d-c), d) & \text{if } m \text{ is odd, } n \text{ even} \end{cases}$$

where m,n are the positive integers corresponding to (a,b) and (c,d), respectively in (4).

It is easy to show that

$$f(m+n) = f(m) \oplus f(n) \qquad \text{and} \qquad f(mn) = f(m) \otimes f(n).$$

A treatment similar to that above for arithmetic and geometric progressions can be found in [1].

REFERENCE

1. M. D. Darkow, "Interpretations of the Peano Postulates," *Amer. Math. Monthly,* Vol. 64, 1957, pp. 270–271.

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A FIBONACCI CURIOSITY

LEON BANKOFF
Los Angeles, California 90048

In the Fibonacci sequence $F_0 = 0,\ F_1 = 1, \cdots, F_n = F_{n-1} + F_{n-2}$,

the sum of the digits of $F_0 = 0$
" " " " " " $F_1 = 1$
" " " " " " $F_5 = 5$
" " " " " " $F_{10} = 10$
" " " " " " $F_{31} = 31$
" " " " " " $F_{35} = 35$
" " " " " " $F_{62} = 62$
" " " " " " $F_{72} = 72$

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