A DENSITY RELATIONSHIP BETWEEN $ax + b$ AND $[x/c]$

ROBERT W. STRINGALL
University of California, Davis, California 95615

This note is motivated by the following problem originating in combinatorial logic. Let f and g be the functions on the set of positive integers defined by $f(x) = 3x$ and $g(x) = [x/2]$, where $[r]$ denotes the greatest integer less than or equal to the real number r. Let Γ denote the collection of all composite functions formed by repeated applications of f and g. For which positive integers k does there exist $h \in \Gamma$ such that $h(1) = k$? For example, if f, g and Γ are defined as above, then

$$f(1) = 3, \quad f^2(1) = 9, \quad f^3(1) = 27, \quad gf^3(1) = 13, \quad fgf^3(1) = 39 \quad \text{and} \quad gfgf^3(1) = 19.$$

Thus, given any number from the collection $\{3, 9, 27, 13, 39, 39\}$ there exists an $h \in \Gamma$ such that $h(1)$ is the given number. The following theorem verifies that every positive integer can be obtained in this manner.

Before stating the theorem, the following conventions are adopted. The set of non-negative integers, the set of positive integers and the set of positive real numbers are denoted by N, N^+ and R^+, respectively. If f and g are functions on N to N, then the composite function $g \cdot f$ is defined by $g \cdot f(x) = g(f(x))$ and the functions obtained by repeated applications of f, n-times, will be denoted by f^n. If r is a real number then the greatest integer less than or equal to r is denoted by $[r]$. Finally, two integers a and c are said to be power related provided there exist $m,n \in N^+$ such that $a^m = c^n$.

Theorem 1. Let $a \neq 1$, $c \neq 1$ be positive integers. Let $b \in N$ and let f and g be the functions on N to N defined by $f(x) = ax + b$ and $g(x) = [x/c]$. If a and c are not power related and if $u,v \in N^+$, then there exist $m,n \in N^+$ such that $g^m \cdot f^n(u) = v$.

Using this theorem with $a = 3$, $b = 0$ and $c = 2$ and noting that 2 and 3 are not power related leads to the previously mentioned result.

A related theorem will be proved from which Theorem 1 will follow. Three lemmas will be employed. Indications of proof will be provided for all three.

Lemma 1. Let $a,c \in N^+$, $a \neq 1$, $c \neq 1$. The collection $\{a^n/c^m : m,n \in N\}$ is dense in R^+ if and only if a and c are not power related.

Proof. This result is well known and is generally considered to be folklore; a guide to its proof is given.

Using the continuity of the logarithm and results found on pages 71–75 of [1], the following statements can be shown to be equivalent.

(a) The collection $\{a^n/c^m : n,m \in N\}$ is dense in R^+.

(b) The collection $\{n - m(\log c/\log a) : n,m \in N\} \cap R^+$ is dense in R^+.

(c) The quotient $(\log c/\log a)$ is irrational.

(d) The numbers a abd c are not power related.

Lemma 2. Let a and b be positive integers with the additional property that the collection $\{a^n/c^m : n,m \in N\}$ is a dense subset of R^+. Then if $n_0 \in N^+$, the collection $\{a^n/c^m : n > n_0; n,m \in N\}$ is also a dense subset of R^+.

Proof. The subset $\{(a^n/c^m)^{n_0} : n,m \in N\} \subseteq \{a^n/c^m : n > n_0; n,m \in N\}$ is dense in R^+.

Lemma 3. Let $a,b \in N$, where $a \neq 0$ and $a \neq 1$. If f is defined on N by $f(x) = ax + b$, then

$$f^n(x) = a^n x + \frac{a^n - 1}{a-1}\, b = a^n \left(\frac{(a-1)x + b(1-a^{-n})}{a-1} \right)$$

for all $n \in N^+$.

Proof. A straightforward induction argument establishes the lemma.

Theorem 2. Let a and c be positive integers neither of which is 1. Let $b \in N$. Let f denote the function on N defined by $f(x) = ax + b$. If a and c are not power related, then for all $u \in N^+$, the collection

$$A(u) = \left\{ \frac{f^n(u)}{c^m} : m,n \in N \right\}$$

is dense in R^+.

Proof. Let $r \in R^+$ and let $\epsilon > 0$ be given. The quotient

$$\frac{r(a-1)}{(a-1)u + b(1-a^{-n})}$$

decreases as n increases and has limiting value

$$\frac{r(a-1)}{(a-1)u+b}\,,$$

as $n \to \infty$. Choose n_0 such that $n > n_0$ implies

$$\frac{r(a-1)}{(a-1)u+b} + \frac{\frac{\epsilon}{2}(a-1)}{(a-1)u+b} > \frac{r(a-1)}{(a-1)u+b(1-a^{-n})}\,.$$

Then for $n > n_0$,

$$\frac{r(a-1)}{(a-1)u+b(1-a^{-n})} < \frac{(r+(\epsilon/2))(a-1)}{(a-1)u+b} < \frac{(r+\epsilon)(a-1)}{(a-1)u+b} \leqslant \frac{(r+\epsilon)(a-1)}{(a-1)u+b(1-a^{-n})}\,.$$

Since a and c are not power related, Lemma 1 yields the fact that $\{a^n/c^m : m,n \in N\}$ is a dense subset of R^+. By Lemma 2, it is possible to choose m_1, n_1 such that $n_1 > n_0$ and

$$\frac{(r+(\epsilon/2))(a-1)}{(a-1)u+b} < \frac{a^{n_1}}{c^{m_1}} < \frac{(r+\epsilon)(a-1)}{(a-1)u+b}\,.$$

It follows that

$$\frac{r(a-1)}{(a-1)u+b(1-a^{-n_1})} < \frac{a^{n_1}}{c^{m_1}} < \frac{(r+\epsilon)(a-1)}{(a-1)u+b(1-a^{-n_1})}$$

and

$$r < \frac{a^{n_1}}{c^{m_1}}\,\frac{(a-1)u+b(1-a^{-n_1})}{a-1} < r+\epsilon\,.$$

By Lemma 3,

$$r < \frac{f^{n_1}(u)}{c^{m_1}} < r+\epsilon\,.$$

Hence $A(u)$ is dense in R^+.

An additional lemma will expedite the proof of Theorem 1.

Lemma 4. Let $c \in N^+$. Let g be defined on R^+ by $g(x) = [x/c]$. If $v \in N^+$ and if r is a real number such that $vc^n \leqslant r < (v+1)c^n$, then $g^n(r) = v$.

Proof. The proof is by induction on n. If $n = 1$, then $vc \leqslant r \leqslant (v+1)c$ implies $r = vc + s$, where $s \in R^+$ or $s = 0$ and $0 \leqslant s < c$. It follows that

$$g(r) = \left[\frac{vc+s}{c}\right] = \left[v + \frac{s}{c}\right] \quad \text{and} \quad \frac{s}{c} < 1\,.$$

Hence $g(r) = v$. Suppose $g^k(r) = v$ whenever $vc^k \leqslant r < (v+1)c^k$. Suppose, in addition, that $vc^{k+1} \leqslant r_0 < (v+1)c^{k+1}$. Then

$$g^{k+1}(r_0) = g^k\left(\left[\frac{r_0}{c}\right]\right) \quad \text{and} \quad vc^k \leqslant \frac{r_0}{c} < (v+1)c^k\ .$$

It follows that

$$vc^k \leqslant \frac{r_0}{c} < (v+1)c^k\ .$$

Hence by the induction hypothesis

$$g^{k+1}(r_0) = g^k \cdot g(r_0) = g^k\left(\left[\frac{r_0}{c}\right]\right) = v\ .$$

To prove Theorem 1, employ Theorem 2 to obtain positive integers n and m such that

$$v < \frac{f^n(u)}{c^m} < v+1$$

and apply Lemma 4.

REFERENCE

1. Ivan Niven, "Irrational Numbers," *The Carus Mathematical Monographs,* No. 11, published by The Mathematical Association of America.

Continued from page 22. *******

We can add any quantity B to each term:

$$x(a+B)^m + y(b+B)^m + (x+y-2)(ax+by+B)^m = (x+y-2)B^m + y(ax+by+B-b)^m + x(ax+by+B-a)^m$$
(where $m = 1, 2$).

A special case of a Fibonacci-type series is

$$1^m \quad 2^m \quad 3^m \quad \cdots \quad n^m\ .$$

Consider the series when $m = 2$:

(1) $\quad 1 \quad 4 \quad 9 \quad 16 \quad 25 \quad \cdots \quad \cdots$

where

$$F_n = 3(F_{n-1} - F_{n-2}) + F_{n-3}$$

[we obtain our coefficients from Pascal's Triangle], i.e.,

$$(x+3)^2 = 3[(x+2)^2 - (x+1)^2] + x^2\ .$$

I have found by conjecture that

$$1^m - 4^m - 4^m - 4^m + 9^m + 9^m + 9^m - 16^m = -0^m - 12^m - 12^m - 12^m + 7^m + 7^m + 7^m + 15^m$$
(where $m = 1, 2$).

[I hope the reader will accept the strange -0^m for the time being.]

If we express the series (1) above in the form

$$a \quad b \quad 3(c-b)+a \quad \text{etc.,}$$

our multigrade appears as follows

$$a^m - 3b^m + 3c^m - [3(c-b)+a]^m = -0^m - 3(3c-4b+a)^m + 3(2c-3b+a)^m + [3(c-b)]^m$$
(where $m = 1, 2$).

We could, of course, write the above as

$$(x^2)^m - 3[(x+1)^2]^m + 3[(x+2)^2]^m - [3[(x+2)^2 - (x+1)^2] + x^2]^m$$
$$= -0^m - 3[x^2 - 4(x+1)^2 + 3(x+2)^2]^m + 3[x^2 - 3(x+1)^2 - 4(x+2)^2]^m + [3[(x+2)^2 - (x+1)^2]^m$$
(where $m = 1, 2$).

Continued on page 82.