ADVANCED PROBLEMS AND SOLUTIONS

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Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.


It is known that, given $k$ a positive integer, each positive integer $n$ has a unique representation in the form

$$n = \left(\binom{a_k}{k}\right) + \left(\binom{a_{k-1}}{k-1}\right) + \cdots + \left(\binom{a_t}{t}\right),$$

where $t = t(n,k)$, $a_i = a_i(n,k)$, $(i = t, \ldots, k)$, $t > 1$ and, if $k > t$, $a_k > a_{k-1} > \cdots > a_t$. Call such a representation the $k$-binomial representation of $n$.

Show that, if $k > 2$, $n = r + s$, where $r > 1$, $s > 1$ and if the $k$-binomial representations of $r$ and $s$ are

$$r = \left(\binom{b_k}{k}\right) + \left(\binom{b_{k-1}}{k-1}\right) + \cdots + \left(\binom{b_u}{u}\right), \quad s = \left(\binom{c_k}{k}\right) + \left(\binom{c_{k-1}}{k-1}\right) + \cdots + \left(\binom{c_v}{v}\right),$$

then

$$\left(\binom{a_k}{k-1}\right) + \left(\binom{a_{k-1}}{k-2}\right) + \cdots + \left(\binom{a_t}{t-1}\right) \leq \left(\binom{b_k}{k-1}\right) + \left(\binom{b_{k-1}}{k-2}\right) + \cdots + \left(\binom{b_u}{u-1}\right) + \left(\binom{c_k}{k-1}\right) + \left(\binom{c_{k-1}}{k-2}\right) + \cdots + \left(\binom{c_v}{v-1}\right).$$


Show that

$$L_p^2 = 1 \pmod{p^2} = L_p \equiv 1 \pmod{p^2}.$$  

H-263 Proposed by G. Berzsenyi, Lamar University, Beaumont, Texas.

Prove that $L_{2mn}^2 = 4 \pmod{L_m^2}$ for every $n, m = 1, 2, 3, \ldots$.

SOLUTIONS

WFFLE!


Suppose an alphabet, $A = \{x_1, x_2, x_3, \ldots\}$, is given along with a binary connective, $P$ (in prefix form). Define a well-formed formula (wff) as follows: a wff is

1. $x_i$ for $i = 1, 2, 3, \ldots$, or
2. If $A_1$, $A_2$ are wffs, then $PA_1A_2$ is a wff and
3. The only wff's are of the above two types.

A wff of order $n$ is a wff in which the only alphabet symbols are $x_1, x_2, \ldots, x_n$ in that order with each letter occurring exactly once. There is one wff of order 1, namely $x_1$. There is one wff of order 2, namely $Px_1x_2$. There are two wff's of order 3, namely $Px_1x_2x_3$ and $PPx_1x_2x_3$, and there are five wff's of order 4, etc.
Define a sequence \( \{ G_i \}_{i=1}^{\infty} \) as follows: 
\( G_i \) is the number of distinct wff’s of order \( i \).

(a) Find a recurrence relation for \( \{ G_i \}_{i=1}^{\infty} \)

(b) Find a generating function for \( \{ G_i \}_{i=1}^{\infty} \).

Solution by P. Bruckman, University of Illinois at Chicago, Chicago, Illinois.

Let \( F_k \) denote any arbitrary wff of order \( k \). In order to form \( F_n \) \((n = 2, 3, \ldots)\), we need to apply \( P \) to all possible distinct “products” of the form \( F_k F_{n-k} \) \((k = 1, 2, \ldots, n-1)\). Hence, we obtain the recursion:

\[
G_n = \sum_{k=1}^{n-1} G_k G_{n-k}, \quad n = 2, 3, 4, \ldots, \text{ with } G_1 = 1.
\]

The above recursion is the well known relation which yields the Catalan numbers, defined by:

\[
G_{n+1} = \frac{2n}{n+1} G_n; \quad \text{thus, } \{ G_n \} = (1, 1, 2, 5, 14, 42, 132, 429, \ldots).
\]

We shall give a brief derivation of (b) from (a), using generating functions. Let us define the generating function for the \( G_n \)’s:

\[
y = \sum_{n=0}^{\infty} G_{n+1} x^n = \sum_{n=1}^{\infty} G_n x^{n-1};
\]

then

\[
y^2 = \sum_{n=0}^{\infty} x^n \sum_{k=0}^{n} G_{k+1} G_{n-k+1} = \sum_{n=0}^{\infty} x^n \sum_{k=1}^{n+1} G_k G_{n+2-k} = \sum_{n=2}^{\infty} x^{n-2} \sum_{k=1}^{n-1} G_n G_{n-k}
\]

\[
= \sum_{n=2}^{\infty} G_n x^{n-2}.
\]

using (a). Hence,

\[
x y^2 = \sum_{n=2}^{\infty} G_n x^{n-1} = y - G_1 = y - 1
\]

(using (1)). Thus, \( y \) is a solution of the quadratic equation

\[
x y^2 - y + 1 = 0; \quad \text{we note that } y(0) = G_1 = 1.
\]

Solving the quadratic, we obtain two solutions:

\[
y = \frac{1 \pm \sqrt{1 - 4x}}{2x}.
\]

The positive sign must be rejected, since this solution is not defined at \( x = 0 \). Thus, \( y = [1 - (1 - 4x)^{1/2}] / 2x \); it is easy to verify, by L’Hospital’s rule, that \( \lim_{x \to 0} y = 1 \). Expanding this expression by the binomial theorem, or otherwise, we find

\[
y = \frac{1}{2x} - \frac{1}{2x} \sum_{n=0}^{\infty} \left( \frac{n}{2} \right) (4x)^n = -\frac{1}{2x} \sum_{n=1}^{\infty} \left( \frac{n}{2} \right) (4x)^n = 2 \sum_{n=0}^{\infty} \left( \frac{n}{n+1} \right) (-4x)^n.
\]
Comparing coefficients with (1), we have:

\[ G_{n+1} = 2 \left( \frac{3}{n+1} \right) (-4)^n = 2 \sqrt{2} \cdot \frac{(-\sqrt{2})}{n+1} \cdot (-4)^n = \frac{2n}{n+1}, \]

which establishes (b).

Also solved by A. Shannon, R. L. Goodstein, and the Proposer.

**SUM DIFFERENTIAL EQUATION!**

**H-235** Proposed by G. Wulczyn, Bucknell University, Lewisburg, Pennsylvania.

a. Find the second-order ordinary differential equation whose power series solution is

\[ \sum_{n=0}^{\infty} F_{n+1} x^n. \]

b. Find the second-order ordinary differential equation whose power series solution is

\[ \sum_{n=0}^{\infty} L_{n+1} x^n. \]

Solution by A. G. Shannon, University of New England, Armidale, N.S.W.

Consider

\[ \begin{cases} H_n & : H_n = H_{n-1} + H_{n-2} \\ H_1 & = F_1 \\ H_2 & = F_2 \text{ when } H_1 = H_2 = 1 \\ L_n & = L_n \text{ when } H_1 = 1, H_2 = 3 \end{cases} \]

Then

\[ \begin{align*} y &= \sum_{n=0}^{\infty} H_{n+1} x^n \quad \text{and} \quad y' &= \sum_{n=0}^{\infty} (n+1)H_{n+2} x^n \quad \text{and} \quad y'' &= \sum_{n=0}^{\infty} (n+1)(n+2)H_{n+3} x^n. \end{align*} \]

Thus

\[ \begin{align*} (x^2 + x - 1)y'' + 2(2x + 1)y' + 2y &= \sum_{n=0}^{\infty} n(n-1)H_{n+1} x^n + \sum_{n=1}^{\infty} n(n+1)H_{n+2} x^n - \sum_{n=0}^{\infty} (n+1)(n+2)H_{n+3} x^n \\ &+ 4 \sum_{n=1}^{\infty} nH_{n+1} x^n + 2 \sum_{n=1}^{\infty} (n+1)H_{n+2} x^n + 2 \sum_{n=0}^{\infty} H_{n+1} x^n \\ &= \sum_{n=2}^{\infty} (n^2 + 3n + 2)(H_{n+1} + H_{n+2} - H_{n+3}) x^n. \end{align*} \]

Thus

\[ (x^2 + x - 1)y'' + 2(2x + 1)y' + 2y = 0 \quad \text{for all} \quad \{ H_n \}. \]

Also solved by P. Bruckman, F. O. Parker, and the Proposer.

**SUM PRODUCT!**

**H-236** Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Show that
where \((x)_k = (1-x)(1-x^3)\cdots (1-x^k), (x)_n = 1.

Solution by P. Bruckman, University of Illinois at Chicago, Chicago, Illinois.

We begin with the well known Jacobi "triple-product" formula:

\[
(1) \quad \prod_{r=1}^{\infty} (1+x^{2r-1}w)(1+x^{2r-1}w^{-1})(1-x^{2r}) = \sum_{n=-\infty}^{\infty} x^n w^n = 1 + \sum_{n=1}^{\infty} x^n (w^n + w^{-n});
\]

the following treatment is formal, and avoids questions of convergence, but it may be shown that the manipulations are valid in the unit disk \(|x| < 1\). Setting \(w = -1\) in (1), the left-hand side becomes:

\[
\prod_{r=1}^{\infty} (1-x^{2r-1})^2(1-x^{2r}) = \prod_{r=1}^{\infty} (1-x^{2r-1})(1-x^{2r}) = \prod_{r=1}^{\infty} \frac{(1-x^{2r-1})(1-x^{2r})}{(1+x^r)}
\]

Hence, we obtain the identity

\[
(2) \quad \prod_{r=1}^{\infty} \left( \frac{1-x^r}{1+x^r} \right) = -1 + 2 \sum_{n=0}^{\infty} (-1)^n x^n = 1 - 2 \sum_{n=0}^{\infty} (-1)^n x^{(n+1)^2}
\]

Next, we will establish the following identity:

\[
(3) \quad \prod_{r=1}^{\infty} (1-x^r w)^{-1} = \sum_{n=0}^{\infty} x^n (x)_n
\]

where \((x)_0 = 1, (x)_n = (1-x)(1-x^3)\cdots (1-x^n), n = 1, 2, 3, \cdots\).

Letting
\[
t(w,x) = \prod_{r=1}^{\infty} (1-x^r w)^{-1} = \sum_{n=0}^{\infty} A_n(x) w^n,
\]

we first note that \(t(0,x) = 1 = A_0(x)\); also, we observe that
\[
t(wx,x) = \prod_{r=1}^{\infty} (1-x^{r+1} w)^{-1} = \prod_{r=2}^{\infty} (1-x^r w)^{-1} = (1-xw)t(w,x).
\]

Hence, by substituting into the series form, we obtain the recursion:
\[
x^n A_n(x) = A_n(x) - xA_{n-1}(x), \quad n = 1, 2, 3, \ldots, \quad A_0(x) = 1,
\]

i.e.,
\[
A_n(x) = \frac{x/ (1-x^n)}{A_{n-1}(x)}.
\]

By an easy induction, we establish that \(A_n(x) = x^n/(x)_n\) for all \(n\), where \((x)_n\) is defined in (3). This establishes (3).

If, in (3), we replace \(w\) by \(-w\), we obtain:
Adding, and then subtracting, both sides of (3) and (4), we obtain:

\[ \prod_{r=1}^{\infty} \left(1 - x^r w\right)^{-1} + \prod_{r=1}^{\infty} \left(1 + x^r w\right)^{-1} = 2 \sum_{n=0}^{\infty} \frac{x^{2n} w^{2n}}{(k)_{2n}}, \]

and

\[ \prod_{r=1}^{\infty} \left(1 - x^r w\right)^{-1} - \prod_{r=1}^{\infty} \left(1 + x^r w\right)^{-1} = 2 \sum_{n=0}^{\infty} \frac{x^{2n+1} w^{2n+1}}{(k)_{2n+1}}. \]

If, in (5) and (6), we set \( w = 1 \), and multiply throughout by

\[ \prod_{r=1}^{\infty} \left(1 - x^r\right), \]

we obtain:

\[ 1 + \sum_{n=0}^{\infty} \frac{x^{2n}}{(k)_{2n}} \prod_{r=1}^{\infty} \left(1 - x^r\right), \]

and

\[ 1 - \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(k)_{2n+1}} \prod_{r=1}^{\infty} \left(1 - x^r\right). \]

Now if, in (7) and (8), we substitute the expression obtained in (2) for the infinite product on the left-hand side, simplify and divide by 2, we obtain the desired results:

\[ \sum_{n=0}^{\infty} (-1)^n x^n = \sum_{n=0}^{\infty} \frac{x^{2n}}{(k)_{2n}} \prod_{r=1}^{\infty} \left(1 - x^r\right), \]

and

\[ \sum_{n=0}^{\infty} (-1)^n x^{n+1} = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(k)_{2n+1}} \prod_{r=1}^{\infty} \left(1 - x^r\right). \]

Also solved by P. Tracy and the Proposer.

\[ \text{SUM RECIPROCAL!} \]


Prove

\[ \sum_{k=0}^{\infty} \frac{1}{F_k^2} = \frac{7 - \sqrt{5}}{2}. \]

Editorial Note: A solution to this problem appears in a note (accepted Feb. 27, 1973) appearing in the Dec. '74 issue of the Quarterly, p. 346.

Solution by A. G. Shannon, University of New England, Arinidale, N.S.W.

Let

\[ F(x) = \sum_{k=1}^{\infty} \frac{x^{k-1}}{F_k^2}. \]
Then
\[ F(ax) = \sum_{k=1}^{\infty} \frac{a^k}{2^k} x^{2^k-1} \]
and so
\[ F(ax) + F(\beta x) = \sum_{k=0}^{\infty} \frac{a^{2^k-1} + \beta^{2^k-1}}{2^k} x^{2^k-1} = \sum_{k=1}^{\infty} \frac{x^{2^k-1}}{2^{k-1}} = \sum_{k=0}^{\infty} \frac{x^{2^k}}{2^k} = x + F(x^2). \]
So
\[ x + F(x^2) = F(ax) + F(\beta x). \]
Put \( x = -\beta : \)
\[ -\beta + F(\beta^2) = F(-\beta^3) + F(-\alpha \beta) \]
or
\[ F(1) = -\beta + 2\beta^3 \]
since
\[ \alpha \beta = -1 \]
and
\[ F(\beta^3) = F(-\beta^3) + 2\beta^3. \]
Thus
\[ \sum_{k=0}^{\infty} \frac{1}{2^k} = 1 + F(1) \]
\[ = 1 - \beta + 2\beta^3 \]
\[ = 2 - (1 + \beta - \beta^3) + \beta \]
\[ = 2 + \frac{(3 - \sqrt{5})}{2} = \frac{7 - \sqrt{5}}{2}. \]

Also solved by I. J. Good (see note), J. Shallit, W. Brady, P. Bruckman, F. Higgins, L. Carlitz, and the Proposer.

**Editorial Note.** Kurt Mahler reports
\[ \sum_{n=0}^{\infty} \frac{1}{2^n} \]
is transcendental.

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