ARITHMETIC SEQUENCES OF HIGHER ORDER

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Definition 1. Given a sequence of numbers

\[ a_0 \quad a_1 \quad a_2 \quad \ldots \quad a_n \quad \ldots \]

we call first differences of (1) the numbers of the sequence

\[ D_0^1 \quad D_1^1 \quad D_2^1 \quad \ldots \quad D_n^1 \quad \ldots \]

with

\[ D_n^1 = a_{n+1} - a_n. \]

By recurrence we define the differences of order k of (1) as the first differences of the sequence of differences of order \( k - 1 \) of (1), namely the numbers of the sequence

\[ D_0^k \quad D_1^k \quad D_2^k \quad \ldots \quad D_n^k \quad \ldots \]

with

\[ D_n^k = D_{n+1}^{k-1} - D_n^{k-1}. \]

Observe that (3) is also valid for \( k = 1 \) if we rename \( a_n = D_n^0 \).

Definition 2. The sequence (1) is arithmetic of order \( k \) if the differences of order \( k \) are equal, whereas the differences of order \( k - 1 \) are not equal. It follows that the differences of order higher than \( k \) are null.

Proposition 1. Given a sequence (1), if there exists a polynomial \( p(x) \) of degree \( k \) with leading coefficient \( c \) such that \( a_n = p(n) \) for \( n = 0, 1, 2, \ldots \) then the sequence is arithmetic of order \( k \) and the differences of order \( k \) are equal to \( kc \).

Proof. Let \( p(x) = cx^k + bx^{k-1} + \ldots \) (the terms omitted are always of less degree than those written). Then

\[ a_n = cn^k + bn^{k-1} + \ldots \]

hence

\[ D_n^1 = a_{n+1} - a_n = c[(n+1)^k - n^k] + b[(n+1)^{k-1} - n^{k-1}] + \ldots = ckn^k + \ldots \]

therefore, for the first differences we have a polynomial \( p_1(x) = kcx^{k-1} + \ldots \) of degree \( k - 1 \) and leading coefficient \( kc \) such that \( D_n^1 = p_1(n) \). Repeating the same process \( k \) times we come to the conclusion that \( D_n^k = p_k(n) \) for a polynomial \( p_k(x) \) of degree zero and leading coefficient \( kc \); hence \( D_n^k = kc \) for \( n = 0, 1, 2, \ldots \).

Example. The sequence

\[ 0 \quad 1 \quad 2^k \quad 3^k \quad \ldots \quad n^k \quad \ldots \]

for \( k \) a positive integer is arithmetic of order \( k \) and \( D_n^k = k! \).

Proposition 2. For any sequence (1), arithmetic or not, we have

\[ D_n^k = \binom{k}{0} a_{n+k} - \binom{k}{1} a_{n+k-1} + \binom{k}{2} a_{n+k-2} + \ldots + \binom{k}{k} a_n. \]

The proof is straightforward using induction on \( k \) with the help of (3).

In particular for the sequence (4) we have
(5) \[ D_n^k = \binom{k}{0} (n+k)^k - \binom{k}{1} (n+k-1)^k + \binom{k}{2} (n+k-2)^k - \ldots + \binom{k}{k} n^k, \]

where the coefficient of \( n^{k-i} \) \((i = 0, 1, 2, \ldots, k)\) is

\[ \left( \binom{k}{i} \right) k^i - \left( \binom{k}{i+1} \right) (k-1)^i + \left( \binom{k}{i+2} \right) (k-2)^i - \ldots \pm \left( \binom{k}{k-1} \right) \binom{k}{i} \binom{k}{i} 0^i \]

(we assume that \( D_i^i = 0 \) for \( i = 1, 2, \ldots, k \) and \( D_0^0 = 1 \)). Hence the coefficient of \( n^{k-i} \) \((i = 1, 2, \ldots, k)\) in (5) is

\[ \left( \binom{k}{i} \right) \left[ \binom{k}{0} k^i - \binom{k}{1} (k-1)^i + \binom{k}{2} (k-2)^i - \ldots + \binom{k}{k-1} \right] 1^i \]

and the coefficient of \( n^k \)

\[ \left( \binom{k}{0} \right) - \left( \binom{k}{1} \right) + \left( \binom{k}{2} \right) - \ldots + \left( \binom{k}{k} \right). \]

Since we know that \( D_n^k = k! \) we have the remarkable equalities:

(i) \[ \left( \binom{k}{0} \right) - \left( \binom{k}{1} \right) + \left( \binom{k}{2} \right) - \ldots + \left( \binom{k}{k} \right) = 0 \]

(which is a very well known fact since it is the development of \((1-1)^k\)).

(6) (ii) \[ \left( \binom{k}{0} \right) k^i - \left( \binom{k}{1} \right) (k-1)^i + \left( \binom{k}{2} \right) (k-2)^i - \ldots + \left( \binom{k}{k-1} \right) 1^i = 0 \]

for \( i = 1, 2, \ldots, k-1 \)

(7) (iii) \[ \left( \binom{k}{0} \right) k^k - \left( \binom{k}{1} \right) (k-1)^k + \left( \binom{k}{2} \right) (k-2)^k - \ldots + \left( \binom{k}{k-1} \right) 1^k = k! \]

A fourth identity can be obtained from (5) with \( n = 0 \) and (21), namely

\[ \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} (k-1-j)^k = (k-1)! \binom{k}{2}. \]

which can also be written in the form

(iv) \[ \left( \binom{k}{0} \right) k^{k+j} - \left( \binom{k}{1} \right) (k-1)^{k+j} + \left( \binom{k}{2} \right) (k-2)^{k+j} - \ldots + \left( \binom{k}{k-1} \right) 1^{k+j} = k! \binom{k+j}{2}. \]

II

Starting with \( k + 1 \) numbers \( A_0, A_1, \ldots, A_k \) we form the “generalized” triangle of Pascal

\[
\begin{array}{cccccccc}
& & & & & & & A_k \\
& & & & & & A_1 & A_2 & \ldots & A_k \\
& & & & & A_0 & A_1 & A_2 & \ldots & A_k \\
A_0 & 2A_0 & + A_1 & A_2 & + 2A_1 & + A_2 & \ldots & A_k \\
3A_0 & 3A_0 & + 3A_1 & + A_2 & A_3 & + 2A_2 & + A_3 & \ldots & A_k \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}
\]

where each number is the sum of the two above. We observe that the coefficient of \( A_o \) in the \( h^{th} \) entry of the \( n^{th} \) row is \( \binom{n-1}{h-1} \); the coefficient of \( A_i \) is \( \binom{n-1}{h-2} \) and the coefficient of \( A_k \) is \( \binom{n-1}{h-k-1} \). (We set \( \binom{n}{j} = 0 \) whenever \( j > n \) or \( j < 0 \).) Therefore the \( h^{th} \) entry of the \( n^{th} \) row is

\[ \binom{n-1}{h-1} A_0 + \binom{n-1}{h-2} A_1 + \ldots + \binom{n-1}{h-k-1} A_k \]

In particular, for the triangle over the \( k + 1 \) differences \( A_0, D_1^1, D_2^1, \ldots, D_k^1 \) of the sequence (1) assumed to be arithmetic of order \( k \), in view of (3) and taking into account that \( D_n^k = D_k^k = \ldots \) we have
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\[ a_0 \ D_0^1 \ D_2^1 \ ... \ D_n^1 \]
\[ a_0 \ a_1 \ D_0^2 \ D_2^2 \ ... \ D_n^2 \]
\[ a_0 \ S_0^1 \ a_1 \ D_0^3 \ D_2^3 \ ... \ D_n^3 \]
\[ a_0 \ S_0^2 \ S_1^2 \ a_1 \ D_0^4 \ D_2^4 \ ... \ D_n^4 \]

where
\[ S_0^k = a_0 \ \ S_1^k = S_{n-1}^k + a_i \ \ \text{and} \ \ S_n^k = S_{n-1}^k + a_{k-1} \]

Since in this triangle \( a_n \) is the \((n+1)\)th entry of the \((n+1)\)th row, we have
\[ a_n = \binom{n}{0} a_0 + \binom{n}{1} D_0^1 + \binom{n}{2} D_1^2 + \binom{n}{3} D_2^3 + \binom{n}{4} D_3^4 \]
or, equivalently,
\[ a_n = \binom{n}{1} D_0^1 + \binom{n}{2} D_1^2 + \binom{n}{3} D_2^3 + \binom{n}{4} D_3^4 \]

Observe that if the sequence (1) is not arithmetic we still can construct a "generalized" triangle of Pascal starting with an infinity of entries in the first row.
\[ a_0 \ D_0^1 \ D_1^2 \ ... \ D_0^n \]

and then instead of (10) we would have
\[ a_n = \binom{n}{1} D_0^1 + \binom{n}{2} D_1^2 + \binom{n}{3} D_2^3 + \binom{n}{4} D_3^4 \]

Proposition 2. If (1) is an arithmetic sequence of order \( k \), we can find a polynomial \( p(x) \) of degree \( k \) such that
\[ a_n = p(n) \].

**Proof.**
\[ p(x) = \binom{n}{0} a_0 + \binom{n}{1} D_0^1 + \binom{n}{2} D_1^2 + \binom{n}{3} D_2^3 + \binom{n}{4} D_3^4 \]

with
\[ \binom{x}{i} = \frac{x(x-1)...(x-i+1)}{i!} \]

is obviously a polynomial of degree \( k \) and in view of (10), \( a_n = p(n) \).

For the partial sum \( S_n = a_0 + a_1 + ... + a_n \) we have a formula similar to (10). In fact, observing that \( S_n \) is the \((n+1)\)th entry of the \((n+2)\)th row in the "generalized" triangle of Pascal, we have
\[ S_n = \binom{n+1}{0} a_0 + \binom{n+1}{1} D_0^1 + \binom{n+1}{2} D_1^2 + \binom{n+1}{3} D_2^3 + \binom{n+1}{4} D_3^4 \]
or, equivalently,
\[ S_n = \binom{n+1}{1} a_0 + \binom{n+1}{2} D_0^1 + \binom{n+1}{3} D_1^2 + \binom{n+1}{4} D_2^3 \]

Therefore \( S_n = p(n) \), where \( p(x) \) is a polynomial of degree \( k + 1 \). This was to be expected, since obviously the sequence \( S_1, S_2, ..., S_n, ... \) is arithmetic of order \( k + 1 \).

**EXAMPLES.** If we apply (11) to the sequences of type (4) with \( k = 1, 2, 3, 4 \) we obtain the well known formulas

1. \[ 0 + 1 + 2 + ... + n = \binom{n+1}{1} 0 + \binom{n+1}{2} 1 = \frac{n^2 + n}{2} \]
2. \[ 0 + 1^2 + 2^2 + ... + n^2 = \binom{n+1}{1} 0 + \binom{n+1}{2} 1 + \binom{n+1}{3} 2 = \frac{n(n+1)(2n+1)}{6} \]
3. \[ 0 + 1^3 + 2^3 + ... + n^3 = \binom{n+1}{1} 0 + \binom{n+1}{2} 1 + \binom{n+1}{3} 6 + \binom{n+1}{4} 1 = \frac{n^4 + 2n^3 + n^2}{4} \]
We now know that the sum
\[ S_k(n) = 0 + 1^k + 2^k + \ldots + n^k \]
is given by a polynomial in \( n \) of degree \( k + 1 \). The question arises, how to find out the coefficients of this polynomial? Obviously the coefficient of \( n^n \) is zero, since \( S_k(0) = 0 \), and the coefficient of \( n^{k+1} \) is \( 1/(k+1) \) as we can see from (11). Hence the polynomial form for \( S_k(n) \) is
\[ S_k(n) = 1/(k+1)n^{k+1} + h_0n^k + h_1n^{k-1} + \ldots + h_{k-1}n + h_0 \]
for some coefficients \( h_0, h_1, \ldots, h_{k-1} \). Since \( S_k(n) - S_k(n-1) = n^k \), we have
\[ \frac{1}{k+1} [n^{k+1} - (n-1)^{k+1}] + h_0[n^k - (n-1)^k] + h_1[n^{k-1} - (n-1)^{k-1}] + \ldots + h_{k-1}n - (n-1)^{k-1}) + h_k = n^k \]
and taking coefficients of the different powers of \( n \), we have the following equations: (the first is an identity, the rest form a linear system of \( k \) equations in \( k \) unknowns, which permits to compute recursively \( h_0, h_1, \ldots, h_{k+1} \).

\[
\begin{align*}
\frac{1}{k+1} \binom{k+1}{1} &= 1, \\
\frac{1}{k+1} \binom{k+1}{2} - \binom{k+1}{1} h_0 &= 0, \\
\frac{1}{k+1} \binom{k+1}{3} - \binom{k+1}{2} h_0 + \binom{k-1}{1} h_1 &= 0, \\
&\quad \vdots \\
\frac{1}{k+1} \binom{k+1}{k} - \binom{k+1}{k-1} h_0 + \binom{k+1}{k-2} h_1 - \ldots - \binom{k+1}{1} h_{k-1} &= 0.
\end{align*}
\]

From the second equation we obtain \( h_0 = \frac{k}{k+1} \), independent of \( k \). If we set
\[ h_1 = \binom{k}{1} b_1, \quad h_2 = \binom{k}{2} b_2 \ldots h_{k-1} = \binom{k}{k-1} b_{k-1} \]
and observe that
\[ \binom{k-i}{j} h_j = \binom{k-i}{j} \binom{k}{j} b_j = \binom{k}{j} \binom{j}{i} b_j \]
we can write the \( j^{th} \) equation in (13) in the form
\[ \frac{1}{i+1} \binom{i}{j} - \binom{i}{j+1} b_1 - \binom{i}{j} b_2 - \ldots - \binom{i}{j} b_i = 0 \]
or, equivalently:
\[ \frac{1}{i+1} - \frac{1}{i+2} (i+1) b_1 - \frac{i+1}{i+2} (i+2) b_2 - \ldots - \frac{i+1}{i+2} (i+1) b_i = 0. \]

Hence the system (13), after omitting the first two identities reduces to:
We will call Bernoulli numbers the numbers \( b_1, b_2, \ldots \). The Bernoulli numbers have over the numbers \( h_1, h_2, \ldots \) the advantage that they do not depend on \( k \), as we can see from system (15). Equation (14) permits to calculate for each \( k \) the \( h \)'s in terms of the \( b \)'s.

**Proposition.** The even Bernoulli numbers are null.

**Proof:** Writing \( n = 1 \) in (12) we have

\[
\frac{1}{k+1} + \frac{1}{2} + h_1 + h_2 + \cdots + h_{k-1} = 1.
\]

On the other hand, the last equation in (13) is

\[
\frac{1}{k+1} - \frac{1}{2} + h_1 - h_2 + \cdots - h_{k-1} = 0.
\]

Adding and subtracting these two equations, we obtain:

\[
\begin{align*}
\frac{1}{k+1} + h_1 + h_3 + \cdots &= \frac{1}{2} - \frac{1}{k+1} \\
h_2 + h_4 + \cdots &= 0
\end{align*}
\]

The second equation in (16) can be written

\[
\left( \frac{k}{2} \right) b_2 + \left( \frac{k}{4} \right) b_4 + \cdots = 0,
\]

where the sum is extended to all the subscripts less than or equal to \( k - 1 \). For \( k = 3 \) we get \( b_2 = 0 \); for \( k = 5, b_4 = 0 \), etc., which proves the proposition.

The first equation in (16) for \( k = 3, 5, 7, \ldots \) yields the infinite system of equations:

\[
\begin{align*}
\left( \frac{3}{1} \right) b_1 &= \frac{1}{2} - \frac{1}{4} \\
\left( \frac{5}{1} \right) b_1 + \left( \frac{5}{3} \right) b_3 &= \frac{1}{2} - \frac{1}{6} \\
\left( \frac{7}{1} \right) b_1 + \left( \frac{7}{3} \right) b_3 + \left( \frac{7}{5} \right) b_5 &= \frac{1}{2} - \frac{1}{8}
\end{align*}
\]

and for \( k = 2, 4, 6, \ldots \) the system

\[
\begin{align*}
\left( \frac{2}{1} \right) b_1 &= \frac{1}{2} - \frac{1}{3} \\
\left( \frac{4}{1} \right) b_1 + \left( \frac{4}{3} \right) b_3 &= \frac{1}{2} - \frac{1}{5} \\
\left( \frac{6}{1} \right) b_1 + \left( \frac{6}{3} \right) b_3 + \left( \frac{6}{5} \right) b_5 &= \frac{1}{2} - \frac{1}{7}
\end{align*}
\]
Subtracting the equations in (18) from those in (17), we have

\[
\begin{align*}
\left( \begin{array}{c} 2 \\ 4 \\ 6 \\ 0 \\ 0 
\end{array} \right) b_1 + \left( \begin{array}{c} 4 \\ 2 \\ 4 \\ 4 \\ 0 
\end{array} \right) b_3 &= \frac{1}{5} \\
\left( \begin{array}{c} 6 \\ 2 \\ 4 \\ 4 \\ 0 
\end{array} \right) b_1 + \left( \begin{array}{c} 6 \\ 4 \\ 4 \\ 4 \\ 4 
\end{array} \right) b_5 &= \frac{1}{7}.
\end{align*}
\]

Any of the infinite systems (17), (18) or (19) permits to find the Bernoulli numbers with odd subscripts.

Substituting in (12) the Bernoulli numbers, we express

\[ S_k(n) = 0 + 1^k + \ldots + n^k \]

in the form

\[ S_k(n) = \frac{1}{k+1} n^{k+1} + \frac{1}{2} n^k + b_1 \binom{k}{1} n^{k-1} + b_3 \binom{k}{3} n^{k-3} + \ldots, \]

where the coefficients of the different powers of \( n \) are products of a combinatorial number of \( k \) and a number which does not depend on \( k \).

**NOTE.** If we compute the coefficient of the \( k^{th} \) power of \( n \) in (11) we have

\[ - \frac{(k+1)(k-2)}{2(k+1)} D_k^5 - \frac{1}{k!} D_k^{k-1}. \]

On the other hand for the sequence \( 0, 1^k, 2^k, \ldots \) that coefficient is \( \frac{1}{k} \), and \( D_k^k = k! \). Hence, for this particular sequence we have

\[ 2D_k^{k-1} = (k-1)k!. \]

**EXAMPLES.** From (10) we obtain:

\[ b_1 = \frac{1}{12}, \quad b_3 = -\frac{1}{120}, \quad b_5 = -\frac{1}{262}, \quad b_7 = -\frac{1}{240}, \quad b_9 = \frac{1}{132} \]

which, substituted in (20) for \( k = 1, 2, \ldots, 11 \) yields the formulas:

\[
\begin{align*}
1 + 2 + \ldots + n &= \frac{1}{2} n^2 + \frac{1}{2} n \\
1^2 + 2^2 + \ldots + n^2 &= \frac{1}{3} n^3 + \frac{1}{2} n^2 + \frac{1}{6} n \\
1^3 + 2^3 + \ldots + n^3 &= \frac{1}{4} n^4 + \frac{1}{2} n^3 + \frac{1}{4} n^2 \\
1^4 + 2^4 + \ldots + n^4 &= \frac{1}{5} n^5 + \frac{1}{2} n^4 + \frac{1}{3} n^3 - \frac{1}{30} n \\
1^5 + 2^5 + \ldots + n^5 &= \frac{1}{6} n^6 + \frac{1}{2} n^5 + \frac{5}{12} n^4 - \frac{1}{12} n^3 \\
1^6 + 2^6 + \ldots + n^6 &= \frac{1}{7} n^7 + \frac{1}{2} n^6 + \frac{1}{2} n^5 - \frac{1}{2} n^3 + \frac{1}{42} n \\
1^7 + 2^7 + \ldots + n^7 &= \frac{1}{8} n^8 + \frac{1}{2} n^7 + \frac{1}{3} n^6 - \frac{1}{12} n^4 + \frac{1}{12} n^3 \\
1^8 + 2^8 + \ldots + n^8 &= \frac{1}{9} n^9 + \frac{1}{2} n^8 + \frac{2}{3} n^7 - \frac{7}{15} n^5 - \frac{2}{9} n^3 - \frac{1}{30} n \\
1^9 + 2^9 + \ldots + n^9 &= \frac{1}{10} n^{10} + \frac{1}{2} n^9 + \frac{3}{4} n^8 - \frac{7}{10} n^6 + \frac{1}{2} n^4 - \frac{3}{20} n^2 \\
1^{10} + 2^{10} + \ldots + n^{10} &= \frac{1}{11} n^{11} + \frac{1}{2} n^{10} + \frac{5}{6} n^9 - n^7 + n^5 - \frac{1}{2} n^3 + \frac{5}{66} n \\
1^{11} + 2^{11} + \ldots + n^{11} &= \frac{1}{12} n^{12} + \frac{1}{2} n^{11} + \frac{11}{12} n^{10} - \frac{11}{8} n^9 + \frac{11}{6} n^8 - \frac{11}{8} n^6 + \frac{5}{12} n^4 \\
\end{align*}
\]