SOME FACTORABLE DETERMINANTS

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A number of computer programs for evaluating determinants of large order are available, however, these programs are quite cumbersome if the determinants are non-symmetric and their order is large. It is rather difficult to test out these computer programs on account of the presence of round-off errors. In many situations, where a researcher is more interested in error assessment, the problem becomes exasperating.

To ease this problem Bowman and Shenton [1] have recently quoted a non-symmetric determinant of order $(s+1)$, given by Painvin [2], which is factorable and have used an ingenious method to show that two other determinants can be reduced to the s^{th} power of a number n, which occurs in the determinant. Since there is only one number n, in each of the determinants, which can be changed arbitrarily the use of these results becomes highly restricted.

We quote below more general forms, containing two arbitrary numbers n and s, of these two factorable determinants. Their proofs are not being given as they are exactly similar to the one given by Bowman and Shenton.

$$
(1)\qquad
\begin{vmatrix}
n-2as & 0 & \cdots & 0 & 0 & 0 & (-2)^s \\
a(2s-1) & n-2a(s-1) & \cdots & \cdot & 0 & 0 & \binom{s}{1}(-2)^{s-1} \\
a(1-s)/2 & a(2s-3) & \cdots & \cdot & \cdot & \cdot & \binom{s}{2}(-2)^{s-2} \\
0 & a(2-s)/2 & \cdots & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdots & n-6a & 0 & 0 & \cdot \\
\cdot & \cdot & \cdots & 5a & n-4a & 0 & \cdot \\
\cdot & \cdot & \cdots & -2a/2 & 3a & n-2a & \binom{s}{s-1}(-2)^1 \\
0 & 0 & \cdots & 0 & -a/2 & a & \binom{s}{s}(-2)^0
\end{vmatrix}
= n^s
$$

It may be noted that there is no "a" in the last column. Each value of a, positive or negative or zero, gives a different determinant, however, the value of the determinant remains unaltered by a and is equal to n^s.

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$$
(2)\qquad
\begin{vmatrix}
n-2as & 0 & 0 & \cdots & 0 & 0 & 0 & (-2)^s \\
2a(s-1) & n-2a(s-1) & 0 & \cdots & \cdot & \cdot & \cdot & \binom{s}{1}(-2)^{s-1} \\
a(1-s)/2 & 2a(s-2) & n-2a(s-2) & \cdots & \cdot & \cdot & \cdot & \binom{s}{2}(-2)^{s-2} \\
0 & a(2-s)/2 & 2a(s-3) & \cdots & \cdot & \cdot & \cdot & \cdot \\
\cdot & 0 & a(3-s)/2 & \cdots & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 0 & \cdots & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdots & n-6a & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdots & 4a & n-4a & 0 & \cdot \\
\cdot & \cdot & \cdot & \cdots & -2a/2 & 2a & n-2a & \cdot \\
0 & 0 & 0 & \cdots & 0 & -a/2 & 0 & \binom{s}{s}(-2)^0
\end{vmatrix}
= (n-2a)^s
$$

The value of the above factorable determinant depends upon the value of a. When n is replaced by $n+2$ and $a=1$, the above result becomes identical with Bowman and Shenton's result.

We also give here a more general form of Painvin's factorable determinant. For all values of n and a, taking either the upper sign or the lower sign at all places, the value of the determinant is $(n+as/2)^{s+1}$.

$$
(3)\qquad
\begin{vmatrix}
n & \pm a/2 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\mp as/2 & n+a & \pm 2a/2 & 0 & \cdots & \cdot & \cdot & 0 \\
0 & \mp a(s-1)/2 & n+2a & \pm 3a/2 & \cdots & \cdot & \cdot & \cdot \\
0 & 0 & \mp a(s-2)/2 & n+3a & \cdots & \cdot & \cdot & \cdot \\
0 & 0 & 0 & \mp a(s-3)/2 & \cdots & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdots & n+(s-2)a & \pm(s-1)a/2 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdots & \mp 2a/2 & n+(s-1)a & \pm sa/2 \\
0 & 0 & 0 & 0 & \cdots & 0 & \mp a/2 & n+sa
\end{vmatrix}
= \left(n+\frac{as}{2}\right)^{s+1}
$$

Evidently, when $a=-1$, and we take the lower sign, the above reduces to Painvin's result.

Proof. Let r denote the number of the row. If the respective rows are multiplied by $(-1)^{r-1}$, $r=1,2,\cdots,s+1$ and added into the first row, then $(n+as/2)$ comes out as a common factor leaving $1,-1,\cdots,(-1)^{s-1}$ as the elements. The order of the determinant can be now reduced by unity by multiplying the new first row by $(\mp as/2)$ and subtracting it from the second row.

In the second operation the respective rows are multiplied by $(-1)^{r-1}\binom{r}{1}$, $r=1,2,\cdots,s$ and added to the first row to give another $(n+as/2)$ as a common factor. The order of the determinant can again be reduced by unity by multiplying the new first row by $\mp a(s-1)/2$ and subtracting it from the second row.

In the third operation the respective rows are multiplied by $(-1)^{r-1}\binom{r+1}{2}$, $r=1,2,\cdots,s-1$ and added together to give another factor $(n+as)/2)$ and then reduction of the order follows the above procedure.

Repeating these operations $(s-4)$ times more, one can easily find that the given determinant reduces to

$$
(n+as/2)^{s-1}\begin{vmatrix} n & \pm s^2a/2 \\ \pm a/2 & n+sa \end{vmatrix}
$$

which gives our result.

REFERENCES

1. K. O. Bowman and L. R. Shenton, "Factorable Determinants," *Mathematics Magazine,* 45 (3), 1972, 144–147.
2. L. Painvin, "Sur un certain systeme d'equations lineaires," *Jour. Math. Pures Appl.,* 2, 1858, 41–46.

NOTE: The author offers a reward of $25 for non-trivial generalizations of the three results in (1), (2) and (3).

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