ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-264 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Show that

$$\sum_{i=0}^{m-r} \binom{s+i}{i}\binom{m+n-s-i+1}{n-s} = \sum_{i=0}^{n-s} \binom{r+i}{i}\binom{m+n-r-i+1}{m-r} .$$

H-265 Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, California.

Show that $F_{2^3\cdot 3^{k-1}} \equiv 0 \pmod{3^k}$, where $k \geqslant 1$.

H-266 Proposed by G. Berzsenyi, Lamar University, Beaumont, Texas.

Find all identities of the form

$$\sum_{k=0}^{n} \binom{n}{k} F_{rk} = s^n F_{tn}$$

with positive integral r, s and t.

SOLUTIONS

TRIPLE PLAY

H-238 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Sum the series

$$S = \sum_{m,n,p=0}^{\infty} x^m y^n z^p ,$$

where the summation is restricted to m,n,p such that

$$m \leqslant n+p, \quad n \leqslant p+m, \quad p \leqslant m+n.$$

Solution by D. Russell, Digital Systems Lab, Stanford, California.

If $m+n+p$ is even, then either (1) exactly one of m, n, or p is even, or (2) all of m, n, and p are even. In either case, $m+n-p$ is also even. Let $a = \frac{1}{2}(m+n-p)$, and similarly let $b = \frac{1}{2}(n+p-m)$ and $c = \frac{1}{2}(p+m-n)$; because of the restrictions, all of a, b, c are non-negative. Then $m = a+c$, $n = a+b$, and $p = b+c$, and

$$x^m y^n z^p = x^{a+c} y^{a+b} z^{b+c} = (xy)^a (yz)^b (xz)^c .$$

This is a general term of the generating function

$$T_{\text{even}} = \frac{1}{(1-xy)(1-yz)(1-xz)}$$

and it is easily seen that *all* terms $x^my^nz^p$ of T_{even} satisfy the restrictions and that $m+n+p$ is even.

Consider the terms where $m+n+p$ is odd and $m+n+p \geqslant 3$ (no terms exist with $m+n+p=1$). Either (1) exactly one of m, n, or p is odd, or (2) all of m, n, and p are odd. In either case, in the restriction $m \leqslant n+p$, equality may not hold, since then one side of the relation would be even and the other would be odd. But if $m < n+p$, then $m-1 \leqslant n+p-2$. Let $m' = m-1$, $n' = n-1$, $p' = p-1$. Then $x^{m'}y^{n'}z^{p'}$ satisfies the restrictions and $m'+n'+p'$ is even. The terms

$$x^my^nz^p = (xyz)x^{m'}y^{n'}z^{p'}$$

with $m+n+p$ odd are thus terms of the generating function $T_{\text{odd}} = xyzT_{\text{even}}$ and *all* terms of T_{odd} are easily seen to satisfy the restrictions with $m+n+p$ odd.

Since $m+n+p$ is either odd or even, the sum S is given by

$$S = \frac{1+xyz}{(1-xy)(1-yz)(1-xz)} .$$

Also solved by P. Bruckman, W. Brady, M. Klamkin, O. P. Lossers, A. Shannon, and the Proposer.

FERMAT' INEQUALITY

H-239 Proposed by D. Finkel, Brooklyn, New York.

If a Fermat number $2^{2^n}+1$ is a product of precisely two primes, then it is well known that each prime is of the form $4m+1$ and each has a unique expression as the sum of two integer squares. Let the smaller prime be a^2+b^2, $a > b$; and the larger prime be c^2+d^2, $c > d$. Prove that

$$\left|\frac{c}{a} - \frac{d}{b}\right| \leqslant \frac{1}{100} .$$

Also, given that $2^{2^6}+1 = (274{,}177)(67{,}280{,}421{,}310{,}721)$ and that $274{,}177 = 516^2 + 89^2$, express the 14-digit prime as a sum of two squares.

Solution by the Proposer.

It is well known that

(1) $$(a^2+b^2)(c^2+d^2) = (ac+bd)^2 + (ad-bc)^2 = (ac-bd)^2 + (ad+bc)^2 .$$

Let $c/a = r$ and $d/b = r'$. Then

(2) $$(a^2+b^2)(c^2+d^2) = (a^2r+b^2r')^2 + (abr'-abr)^2 = (a^2r-b^2r')^2 + (abr'+abr)^2 .$$

One of the four squares on the right-hand side of (2) must be 1^2. Taking a, b, c, and d as positive, it is obviously not the first one on the top line or the last one on the bottom line. Clearly $ac > bd$ and thus $(a^2r - b^2r')^2 > 1$. Hence,

$$(abr'-abr)^2 = 1^2 \quad \text{or} \quad |r'-r| = \frac{1}{ab} .$$

The smallest Fermat number which is a product of exactly two primes is $2^{2^5}+1 = (641)(6{,}700{,}417)$. Here $641 = 25^2 + 4^2$ and $ab = 100$. No other Fermat number can have an ab product as low as 100.* Hence the result,

$$\left|\frac{c}{a} - \frac{d}{b}\right| \leqslant \frac{1}{100} ,$$

follows. For the last part of the problem, let the smaller prime be $p_1 = a^2+b^2$ and the larger prime be

$$p_2 = c^2+d^2 = a^2r^2 + b^2(r')^2 .$$

Now r and r' are approximately equal and $p_2/p_1 \sim r^2$. Since $c = ar$ and $d = br'$, a simple calculation leads to

$$p_2 = 8083111^2 + 1394180^2 .$$

*Beiler, *Recreations in the Theory of Numbers,* pp. 143, 175.

E-GAD

H-240 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Let

$$S(m,n,p) = (q)_n(q)_p \sum_{i=0}^{\min(n,p)} \frac{q^{mi+(n-i)(p-i)}}{(q)_i(q)_{n-i}(q)_{p-i}},$$

where

$$(q)_j = (1-q)(1-q^2)\cdots(1-q^j), \qquad (q)_0 = 1.$$

Show that $S(m,n,p)$ is symmetric in m,n,p.

Solution by the Proposer.

Put

$$e(x) = \prod_{n=0}^{\infty}(1-q^nx)^{-1} = \sum_{n=0}^{\infty}\frac{x^n}{(q)_n}.$$

It is well known and easy to show that

$$\sum_{n=0}^{\infty}\frac{(a)_n}{(q)_n}x^n = \frac{e(x)}{e(ax)},$$

where

$$(a)_n = (1-a)(1-qa)\cdots(1-q^{n-1}a), \qquad (a)_0 = 1.$$

It follows that

$$\sum_{r,s=0}^{\infty}\frac{q^{rs}x^ry^s}{(q)_r(q)_s} = \sum_{r=0}^{\infty}\frac{x^r}{(q)_r}\sum_{s=0}^{\infty}\frac{(q^ry)^s}{(q)_s} = \sum_{r=0}^{\infty}\frac{x^r}{(q)_r}e(q^ry) = e(y)\sum_{r=0}^{\infty}\frac{(y)_r}{(q)_r}x^r,$$

so that

$$\sum_{r,s=0}^{\infty}\frac{q^{rs}x^ry^s}{(q)_r(q)_s} = \frac{e(x)e(y)}{e(xy)}.$$

Then

$$\frac{e(x)e(y)e(z)}{e(xyz)} = \frac{e(x)e(yz)}{e(xyz)}\,\frac{e(y)e(z)}{e(yz)}$$

$$= \sum_{m,i=0}^{\infty}\frac{q^{mi}x^m(yz)^i}{(q)_m(q)_i}\sum_{j,k=0}^{\infty}\frac{q^{jk}y^jz^k}{(q)_j(q)_k} = \sum_{m,n,p=0}^{\infty}\frac{x^my^nz^p}{(q)_m}\sum_{\substack{i+j=n\\ i+k=p}}\frac{q^{mi+jk}}{(q)_i(q)_j(q)_k}$$

$$= \sum_{m,n,p=0}^{\infty}\frac{x^my^nz^p}{(q)_m}\sum_{i=0}^{\min(n,p)}\frac{q^{mi+(n-i)(p-i)}}{(q)_i(q)_{n-i}(q)_{p-i}},$$

so that

$$(*) \qquad \frac{e(x)e(y)e(z)}{e(xyz)} = \sum_{m,n,p=0}^{\infty}\frac{x^my^nz^p}{(q)_m(q)_n(q)_p}\,S(m,n,p).$$

The stated result follows at once.

REMARK. Since

$$\frac{1}{e(z)} = \sum_{n=0}^{\infty} (-1)^n \frac{q^{\frac{1}{2}n(n-1)}z^n}{(q)_n},$$

it follows from (*) that

$$\sum_{m,n,p=0}^{\infty} \frac{x^m y^n z^n}{(q)_m (q)_n (q)_p} S(m,n,p) = \sum_{i,j,k=0}^{\infty} \frac{x^i y^j z^k}{(q)_i (q)_j (q)_k} \sum_{s=0}^{\infty} (-1)^s q^{\frac{1}{2}s(s-1)} \frac{(xyz)^s}{(q)_s},$$

so that

$$S(m,n,p) = \sum_{s=0}^{\min(m,n,p)} (-1)^s q^{\frac{1}{2}s(s-1)} \frac{(q)_m (q)_n (q)_p}{(q)_s (q)_{m-s} (q)_{n-s} (q)_{p-s}}.$$

HARMONIC

H-241 Proposed by R. Garfield, College of Insurance, New York, New York.

Prove that

$$\frac{1}{1-x^n} = \frac{1}{n} \sum_{k=0}^{n-1} \frac{1}{1-xe^{\frac{2k\pi}{n}i}}.$$

Solution by G. Wulczyn, Bucknell University, Lewisburg, Pennsylvania.

Using partial fractions, we have

$$\frac{1}{1-x^n} = -\frac{1}{x^n-1} = -\sum_{k=0}^{n-1} \frac{A_k}{x - e^{\frac{2\pi k}{n}i}}$$

$$A_k = \frac{1}{\phi'\left(e^{\frac{2\pi k}{n}i}\right)}; \quad \phi(x) \equiv x^n - 1$$

[e.g., see Edwards, *Integral Calculus*, Vol. 1, p. 145.]

$$\frac{1}{1-x^n} = -\sum_{k=0}^{n-1} \frac{1}{ne^{\frac{2\pi k(n-1)}{n}i}\left(x - e^{\frac{2\pi k}{n}i}\right)} = \frac{1}{n}\sum_{k=0}^{n-1} \frac{1}{1-xe^{\frac{2\pi k(n-1)}{n}}} = \frac{1}{n}\sum_{k=0}^{n-1} \frac{1}{1-xe^{\frac{2\pi k}{n}i}}$$

since $(n, n-1) = 1$.

Also solved by C. Bridger, P. Smith, and the Proposer.

PELL-MELL

H-243 Proposed by G. Wulczyn, Bucknell University, Lewisburg, Pennsylvania.

Show that for each triangular number $t_n = \frac{n}{2}(n+1)$ there exist an infinite number of nonsquare positive integers D such that $t_{n+r}^2 - t_n^2 D = 1$.

Solution by the Proposer.

In the Pellian equation $x^2 - Dy^2 = 1$, let $x = t_{mt_n^2+1}$, $y = t_n$, $mn \neq 0$.

$$[(mt_n^2+1)(mt_n^2+2)+2]\,[(mt_n^2+1)(mt_n^2+2)-2] = 4t_n^2 D.$$

$$[m^2t_n^4 + 3mt_n^2 + 4](m^2t_n^2 + 3m) = 4D.$$

The left-hand side is congruent to zero modulo 4 for the conditions (1) m an even integer, (2) m odd, t_n odd, (3) m odd, t_n even. Hence D is an integer, and not an integer square since the difference of these two integer squares is never one.

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