POLYNOMIALS $P_{2n+1}(x)$ SATISFYING $P_{2n+1}(F_k) = F_{(2n+1)k}$

DAVID G. BEVERAGE
Civilian Instructor, USN Program for Afloat College Education, Chapman College, Orange, Calif. 92666

Denote the polynomials defined indirectly and recursively by my Theorem [1] by $P_{2n+1}(x)$ so that

$$P_1(x) = x, \quad P_3(x) = 5x^3 + 3(-1)^k x, \quad P_5(x) = 25x^5 + 25(-1)^k x^3 + 5x,$$

$$P_7(x) = 125x^7 + 175(-1)^k x^5 + 70x^3 + 7(-1)^k x,$$

$$P_9(x) = 5^4 x^9 + 3^2 \cdot 5^3 (-1)^k x^7 + 3^3 \cdot 5^2 x^5 + 2 \cdot 3 \cdot 5^2 (-1)^k x^3 + 3^2 x,$$

$$P_{11}(x) = 5^5 x^{11} + 5^4 \cdot 11(-1)^k x^9 + 2^2 \cdot 5^3 \cdot 11 x^7 + 5^2 7 \cdot 11(-1)^k x^5 + 5^2 11 x^3 + 11(-1)^k x,$$

$$P_{13}(x) = 5^6 x^{13} + 5^5 13(-1)^k x^{11} + 5^5 13 x^9 + 2^2 \cdot 3 \cdot 5^3 13(-1)^k x^7 + 2 \cdot 5^2 7 \cdot 13 x^5 + 5 \cdot 7 \cdot 13(-1)^k x^3 + 13x,$$

$$P_{15}(x) = 5^7 x^{15} + 3 \cdot 5^7 (-1)^k x^{13} + 2 \cdot 3^2 5^6 x^{11} + 5^6 11(-1)^k x^9 + 2 \cdot 3^2 5^5 x^7 + 2 \cdot 3^3 5^2 7(-1)^k x^5$$
$$+ 2^2 5^2 7 x^3 + 3 \cdot 5(-1)^k x \quad .$$

Theorem [1] may be written as

Theorem 1.

$$(1) \qquad P_{2n+1}(x) = 5^n x^{2n+1} - \sum_{s=1}^{n} \binom{2n+1}{n+1-s} [(-1)^{k+1}]^{n+1-s} P_{2s-1}(x) \; .$$

The following Theorem 2 gives an explicit expression for these polynomials:

Theorem 2.

$$(2) \qquad P_{2n+1}(x) = \sum_{r=0}^{n} 5^{n-r}(-1)^{kr} \frac{(2n+1)[(2n-r)!]}{r!(2n+1-2r)!} x^{2n+1-2r} \; .$$

Proof. The polynomials $P_3(x)$ obtained by substituting $n = 1$ into Eqs. (1) and (2) are easily shown to be identical. Using the second principle of mathematical induction, assume that (1) and (2) express identical polynomials $P_{2s+1}(x)$ for all $s < n$ [2]. Substituting the expression (2) into the right-hand side of Eq. (1), it will be shown that the resulting expression for $P_{2n+1}(x)$ is identical to that as determined by Eq. (2). Thus, Eq. (1) becomes

$$P_{2n+1}(x) = 5^n x^{2n+1} - \sum_{s=1}^{n} \binom{2n+1}{n+1-s} [(-1)^{k+1}]^{n+1-s} P_{2s-1}(x),$$

where

$$P_{2s-1}(x) = \sum_{r=0}^{s-1} 5^{s-1-r} x^{2s-1-2r} (-1)^{kr} \frac{(2s-1)[(2s-2-r)!]}{r!(2s-2r-1)!} \quad .$$

Rearranging terms and changing the variable of summation by $t = s - r - 1$, eliminating r, and interchanging the order of summation on t and s obtain:

(3) $$P_{2n+1}(x) = 5^n x^{2n+1} - \sum_{t=0}^{n-1} 5^t(-1)^{kn-kt+n+1} \frac{(2n+1)!x^{2t+1}}{(2t+1)!} \cdot Q,$$

where

$$Q = \sum_{s=t+1}^{n} (-1)^s \frac{(2s-1)[(s+t-1)!]}{(s-t-1)!(n-s+1)!(n+s)!} .$$

Expression Q may be summed using the antidifference method as

$$Q = \left.\frac{-(-1)^s(s+t-1)!}{(n-t)(n-s+1)!(n+s-1)!(s-t-2)!}\right|_{s=t+1}^{s=n+1} = \frac{-(-1)^{n+1}(n+t)!}{(n-t)(2n)!(n-t-1)!} .$$

Substituting the latter for Q in (3) above and simplifying, obtain

(4) $$P_{2n+1}(x) = 5^n x^{2n+1} + \sum_{t=0}^{n-1} (-1)^{k(n-t)} \frac{5^t(2n+1)x^{2t+1}}{(2t+1)!(n-t)!} (n+t)!$$

Finally, changing the variable of summation to $r = n - t$, and noting the first term is represented with $r = 0$, Eq. (4) becomes (2) as desired.

Theorem 3. $$P_{(2m+1)(2n+1)}(x) = P_{2m+1}(P_{2n+1}(x)) .$$

Proof. Each of the polynomials is of degree $(2m+1)(2n+1)$, and since the same Fibonacci number, namely, $F_{(2m+1)(2n+1)k}$, is obtained for $x = F_k$, $k = 1, 2, 3, \cdots$, the polynomials have identical values for an infinite number of arguments, and thus by a well known property of polynomials, the polynomials in x are identical [3].

Theorem 4.

(5) $$P_{2n+5}(x) = [5x^2 + 2(-1)^k]P_{2n+3}(x) - P_{2n+1}(x) .$$

Proof. Substituting (1) into the right-hand member of Eq. (5) above, and multiplying by $[5x^2 + 2(-1)^k]$, one obtains three summations:

$$\sum_{r=0}^{n+1} 5^{n+2-r}(-1)^{kr} \frac{(2n+3)[(2n+2-r)!]}{r!(2n+3-2r)!} x^{2n+5-2r}$$

$$+\sum_{r=0}^{n+1} 5^{n+1-r}(-1)^{k(r+1)} \frac{2(2n+3)[(2n+2-r)!]}{r!(2n+3-2r)!} x^{2n+3-2r}$$

$$-\sum_{r=0}^{n} 5^{n-r}(-1)^{kr} \frac{(2n+1)[(2n-r)!]}{r!(2n+1-2r)!} x^{2n+1-2r} .$$

Replacing r by $r - 1$ and $r - 2$ in the second and third summations, respectively, each summation has the common factor $5^{n+2-r}(-1)^{kr}x^{2n+5-2r}$ with the range of summation overlapping for $r = 2$ to $n + 1$ as follows:

$$\sum_{r=0}^{n+1} 5^{n+2-r}(-1)^{kr} \frac{(2n+3)[(2n+2-r)!]}{r!(2n+3-2r)!} x^{2n+5-2r}$$

$$+\sum_{r=1}^{n+2} 5^{n+2-r}(-1)^{kr} \frac{2(2n+3)[(2n+3-r)!]}{(r-1)!(2n+5-2r)!} x^{2n+5-2r} -$$

$$-\sum_{r=2}^{n+2} 5^{n+2-r}(-1)^{kr}\,\frac{(2n+1)[(2n+2-r)!]}{(r-2)!(2n+5-2r)!}x^{2n+5-2r}\,.$$

Collecting the overlapping portion of the summations in a single summation and simplifying the remaining individual terms, one obtains:

$$\sum_{r=2}^{n+1} 5^{n+2-r}(-1)^{kr}x^{2n+5-2r}\left\{\frac{(2n+3)[(2n+2-r)!]}{r!(2n+3-2r)!}+\frac{2(2n+3)[(2n+3-r)!]}{(r-1)!(2n+5-2r)!}-\frac{(2n+1)[(2n+2-r)!]}{(r-2)!(2n+5-2r)!}\right\}$$

(6)
$$+5^{n+2}x^{2n+5}+5^{n+1}(-1)^k(2n+5)x^{2n+3}+(2n+5)(-1)^{kn}x\,.$$

The expression within the brace of Eq. (6) becomes

$$\frac{(2n+2-r)![(2n+3)(2n+5-2r)(2n+4-2r)+2r(2n+3)(2n+3-r)-r(r-1)(2n+1)]}{r!(2n+5-2r)!}$$

$$=\frac{(2n+2-r)!}{r!(2n+5-2r)!}\,[8n^3+48n^2+94n-8rn^2-34rn+2r^2n+5r^2-35r+60]$$

$$=\frac{(2n+2-r)!}{r!(2n+5-2r)!}\,[(2n+5)(2n+4-r)(2n+3-r)] = \frac{(2n+5)[(2n+4-r)!]}{r!(2n+5-2r)!}\,.$$

Substituting this simplified result for the brace in (6) and noting the three individual terms in (6) from the summation general summation term with $r = 0, 1,$ and $n+2$, respectively, Eq. (6) becomes $P_{2n+5}(x)$ as expressed by (2).

Theorem 5.

(7)
$$[5x^2+4(-1)^k]P''_{2n+1}(x)+5xP'_{2n+1}(x)-5(2n+1)^2P_{2n+1}(x) = 0\,.$$

Proof. Differentiating (2) and substituting into the left-hand member of (7), multiplying the binomial $[5x^2 + 4(-1)^k]$ appropriately in (7) to form two summations and changing the index of summation r to $r-1$ in the summation formed from $4(-1)^kP''_{2n+1}(x)$, one obtains four summations with like general terms in x with the range of summation overlapping from $r = 1$ to $n-1$. Factoring out the common factors, the left-hand member of (7) becomes

$$\sum_{r=1}^{n-1}\frac{5^{n-r+1}(2n+1)(-1)^{kr}(2n-r)!}{r!\,(2n-2r+1)!}\left\{(2n-2r+1)(2n-2r)+4r(2n-r+1)+(2n+1-2r)-(2n+1)^2\right\}x^{2n+1-2r}$$

$$+\frac{5^{n+1}(2n+1)[(2n)!]x^{2n+1}}{0!\,(2n-1)!}+\frac{5(-1)^{kn}(2n+1)[(n+1)!]4x}{(n-1)!\,1!}+\frac{5^{n+1}(2n+1)[(2n)!]x^{2n+1}}{0!\,(2n)!}+\frac{5(-1)^{kn}(2n+1)[n!]x}{n!\,0!}$$

$$-\frac{5^{n+1}(2n+1)^3[(2n)!]x^{2n+1}}{0!\,(2n+1)!}-\frac{5(-1)^{kn}(2n+1)^3(n!)x}{n!\,1!}\,.$$

The expression within the brace of the summation is easily shown to be zero, and the remaining six individual terms are easily shown to be zero also.

Theorem 6. The polynomials $P_{2n+1}(x)$ satisfy

$$P_{2n+1}(x) = \frac{2}{\sqrt{5}}\,T_{2n+1}\left(\frac{\sqrt{5}}{2}x\right) \quad\text{or}\quad (-1)^n\,\frac{2i}{\sqrt{5}}\,T_{2n+1}\left(-i\,\frac{\sqrt{5}}{2}\,x\right)$$

according to k odd or even, where $T_{2n+1}(x)$ is the Chebyshev polynomial of the first kind [4].

Proof. For k odd,

$$P'_{2n+1}(x) = \frac{2}{\sqrt{5}}\,T'_{2n+1}\left(\frac{\sqrt{5}}{2}x\right)\frac{\sqrt{5}}{2} \quad\text{and}\quad P''_{2n+1}(x) = \frac{2}{\sqrt{5}}\,T''_{2n+1}\left(\frac{\sqrt{5}x}{2}\right)\frac{5}{4}$$

by applying the chain rule.

Substituting into (7) and changing the variable x to z by $x = (2/\sqrt{5})z$, obtain

$$(1-z^2)T''_{2n+1}(z) - z\cdot T'_{2n+1}(z) + (2n+1)^2T_{2n+1}(z) = 0$$

defining the required polynomials [4: 22.6.9 p. 781]. The case for k even may be handled similarly.

REFERENCES

1. David G. Beverage, "A Polynomial Representation of Fibonacci Numbers," *The Fibonacci Quarterly,* Vol. 9, No. 5 (Dec. 1971), pp. 541–544.
2. Nathan Jacobson, *Lectures in Abstract Algebra,* D. Van Nostrand, 1951, Vol. 1, p. 9.
3. L. E. Dickson, *New First Course in the Theory of Equations,* John Wiley & Sons, 1960, p. 15, Th. 4.
4. *Handbook of Mathematical Functions,* U.S. Dept. Commerce, National Bureau of Standards, Applied Math Series 55, pp. 773–795.

****

[Continued from page 196.]

Let $k > 0$, $2|k$, $K := 4k+3$; the conditions

$$v_{k+1}r_k + v_kr_{k+1} = 2^K, \quad 0 < r_{k+1} \leqslant 2v_{k+1}, \quad 2 \nmid r_{k+1}$$

define the integers r_{k+1}, r_k uniquely. Then $2r_{k+1} < r_k$. Let

$$r_j := 2r_{j+1} + r_{j+2} \quad (j = k-1, k-2, \cdots, 1);$$

then

$$0 < 2r_{j+1} < r_j, \quad 2 \nmid r_j \Leftrightarrow 2 \nmid j, \quad v_{j+1}r_j + v_jr_{j+1} = 2^K \quad (j = k-1, k-2, \cdots, 1);$$

$j = 1$ gives

$$2r_1 + r_2 = 2^K, \quad 0 < 2r_1 < 2^K.$$

Let $y_k := 2\cdot 2^K + r_1$, $x_k := 3y_k + 2^K$; then $2\cdot 2^K < y_k$, $2 \nmid y_k$, $2 \nmid x_k$. The defining equation for x_k gives $H(x_k, y_k) = 2$. The defining equations for x_k, y_k, r_j $(j = 1, 2, \cdots, k-1)$ are the beginning of an algorithm by greatest and by nearest integers for x_k, y_k and therefore $N(x_k, y_k) > k$. For an arbitrary integer $s > 0$, let $g_s := x_s$, $h_s := y_s$ in case $2|s$ and $g_s := x_{s+1}$, $h_s := y_{s+1}$ in case $2 \nmid s$. This proves

Theorem 2. For every integer $s > 0$ there exist odd integers $g_s > h_s > 0$ with $E(g_s, h_s) \geqslant N(g_s, h_s) > s$, $H(g_s, h_s) = 2$.

Nothing is known about the average size of $H(a,b)$.

****