A GENERALIZATION OF A SERIES OF DE MORGAN, WITH APPLICATIONS OF FIBONACCI TYPE

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Bromwich [1], p. 24, attributes the formula

(1) $$\frac{x}{1-x^2}+\frac{x^2}{1-x^4}+\frac{x^4}{1-x^8}+\cdots+\frac{x^{2^{n-1}}}{1-x^{2^n}}=\frac{1}{1-x}-\frac{1}{1-x^{2^n}}$$

to Augustus de Morgan, together with the corresponding sums of the infinite series, namely $x(1-x)^{-1}$ if $|x|<1$, and $(1-x)^{-1}$ if $|x|>1$. As far as the authors know, the following generalization has not yet appeared in print.

(2) $$\sum_{n=0}^{N}\frac{(xy)^{m^n}[x^{m^n(m-1)}-y^{m^n(m-1)}]}{[x^{m^n}-y^{m^n}][x^{m^{n+1}}-y^{m^{n+1}}]}=\frac{y}{x-y}-\frac{y^{m^{N+1}}}{x^{m^{N+1}}-y^{m^{N+1}}}\qquad(x\neq y)$$

$$=\frac{x}{x-y}-\frac{x^{m^{N+1}}}{x^{m^{N+1}}-y^{m^{N+1}}}\qquad(m=2,3,\cdots).$$

To see this, note that the expression

$$[z^{m^n}+z^{2\cdot m^n}+\cdots+z^{(m-1)m^n}][1+z^{m^{n+1}}+z^{2m^{n+1}}+\cdots\text{ad inf}]=\frac{z^{m^n}(1-z^{m^n(m-1)})}{(1-z^{m^n})(1-z^{m^{n+1}})}\qquad(|z|<1)$$

is equal to the sum of all those powers of z where the powers are multiples of m^n but not multiples of m^{n+1}. Therefore

(3) $$\sum_{n=0}^{N}\frac{z^{m^n}(1-z^{m^n(m-1)})}{(1-z^{m^n})(1-z^{m^{n+1}})}=\frac{1}{1-z}-\frac{1}{1-z^{m^{N+1}}}\qquad(|z|<1).$$

On replacing z by y/x we obtain (2), and, on allowing N to tend to infinity we obtain, if $|x|\neq|y|$,

(4) $$\sum_{n=0}^{\infty}\frac{(xy)^{m^n}[x^{m^n(m-1)}-y^{m^n(m-1)}]}{(x^{m^n}-y^{m^n})(x^{m^{n+1}}-y^{m^{n+1}})}=\frac{\text{min (abs)}\,(x,y)}{x-y}\qquad(m=2,3,\cdots),$$

where min (abs) (x,y) signifies x or y, depending on whether $|x|<|y|$ or $|x|>|y|$, respectively.

To obtain examples, let a and b be positive integers and let u_n be the denominator of the $(n-1)^{th}$ convergent of the continued fraction

$$\frac{b}{a+}\,\frac{b}{a+}\,\frac{b}{a+}\cdots$$

so that

(5)
$$u_n = \frac{\xi^n - \eta^n}{\sqrt{(a^2+4b)}},$$

where

$$\xi = \frac{a+\sqrt{(a^2+4b)}}{2}, \qquad \eta = \frac{a-\sqrt{(a^2+4b)}}{2}.$$

Now put $x = \xi^k$ and $y = \eta^k$ in (2), where k is a positive integer, and we obtain

(6)
$$\sum_{n=0}^{N} \frac{(-b)^{km^n} u_{km^n(m-1)}}{u_{km^n} u_{km^{n+1}}} = \eta - \frac{\eta^{km^{N+1}}}{u_{km^{N+1}}}.$$

When $b = 1$ the formula simplifies somewhat. When $a = b = 1$, then $u_n = F_n$, the n^{th} Fibonacci number. Some special cases of these formulae are

(7)
$$\frac{2(2+1)}{2^3-1} + \frac{2^3(2^3+1)}{2^9-1} + \frac{2^9(2^9+1)}{2^{27}-1} + \cdots = 1 \quad (x = 2, y = 1, m = 3 \text{ in } (4));$$

(8)
$$\frac{1}{F_1} + \frac{1}{F_2} + \frac{1}{F_4} + \frac{1}{F_8} + \cdots = \frac{7-\sqrt{5}}{2} \quad (a = b = k = 1, m = 2 \text{ in } (6), \text{ as } N \to \infty);$$

(9)
$$\frac{L_1}{F_3} + \frac{L_3}{F_9} + \frac{L_9}{F_{27}} + \cdots = \frac{\sqrt{5}-1}{2} \quad (a = b = k = 1, m = 3 \text{ in } (6), \text{ as } N \to \infty);$$

(where $L_1 = 1, L_2 = 3, L_3 = 4, L_4 = 7, \cdots$ are the Lucas numbers),

(10)
$$\frac{F_1}{L_3} + \frac{F_3}{L_9} + \frac{F_9}{L_{27}} + \cdots = \frac{5-\sqrt{5}}{10},$$

(11)
$$\sum_{n=0}^{\infty} \frac{L_{k3^n}}{F_{k3^{n+1}}} = \frac{(\sqrt{5}-1)^k}{2^k F_k}, \qquad \sum_{n=0}^{\infty} \frac{F_{k3^n}}{L_{k3^{n+1}}} = \frac{(\sqrt{5}-1)^k}{2^k \sqrt{5} L_k}.$$

Further generalization. Formulae (2) and (4) can be further generalized to

(12)
$$\frac{y_0}{x_0 - y_0} = \sum_{n=0}^{m-1} \frac{x_{n+1}y_n - x_n y_{n+1}}{(x_n - y_n)(x_{n+1} - y_{n+1})} + \frac{y_m}{x_m - y_m},$$

where $\{x_n\}$ and $\{y_n\}$ are any sequences of real or complex numbers, with $x_n \neq y_n\ \forall n$, and

(13)
$$\sum_{n=0}^{\infty} \frac{x_{n+1}y_n - x_n y_{n+1}}{(x_n - y_n)(x_{n+1} - y_{n+1})} = \begin{cases} y_0/(x_0 - y_0) & \text{if } y_n/x_n \to 0 \\ x_0/(x_0 - y_0) & \text{if } x_n/y_n \to 0 \end{cases}.$$

Although (12) is more general than (2), its proof is obvious. A special case of (13), after a change of notation, is

(14)
$$\sum_{n=m}^{\infty} \frac{(xy)^{s(n)}[x^{s(n+1)-s(n)} - y^{s(n+1)-s(n)}]}{[x^{s(n)} - y^{s(n)}][x^{s(n+1)} - y^{s(n+1)}]} = \frac{y^{s(m)}}{x^{s(m)} - y^{s(m)}}$$

if $|x| > |y|$ and $s(n) \to \infty$ as $n \to \infty$.

Now put $x = \xi$, $y = \eta$ and we obtain

(15)
$$\sum_{n=m}^{\infty} \frac{(-b)^{s(n)} u_{s(n+1)-s(n)}}{u_{s(n)} u_{s(n+1)}} = \frac{\eta^{s(m)}}{u_{s(m)}}$$

and in particular

(16) $$\sum_{n=0}^{\infty} \frac{(-1)^{s(n)} F_{s(n+1)-s(n)}}{F_{s(n)}F_{s(n+1)}} = \left(\frac{1-\sqrt{5}}{2}\right)^{s(0)} \Big/ F_{s(0)}\,.$$

For example, if $s(n) = F_{n+1}$,

(17) $$\sum_{n=1}^{\infty} \frac{\epsilon_n F_{F_n}}{F_{F_{n+1}}F_{F_{n+2}}} = \frac{\sqrt{5}-1}{2}, \quad \text{where} \quad \epsilon_n = \begin{cases} 1 & \text{if } n \equiv 0 \text{ or } 1 \pmod 3 \\ -1 & \text{if } n \equiv 2 \pmod 3 \end{cases}$$

and if $s(n) = L_{n+1}$,

(18) $$\sum_{n=0}^{\infty} \frac{\epsilon_n F_{L_n}}{F_{L_{n+1}}F_{L_{n+2}}} = \frac{\sqrt{5}-1}{2}\,.$$

Putting $s(n) = (n+1)k$ in (16) gives

(19) $$\sum_{n=0}^{\infty} \frac{(-1)^{(n+1)k}}{F_{(n+1)k}F_{(n+2)k}} = \frac{\left(\frac{1-\sqrt{5}}{2}\right)^k}{F_k^2}\,.$$

Putting $x_n = (n+2)^c$ and $y_n = 1$ in (13) gives, after a change of notation,

(20) $$\sum_{n=2}^{\infty} \frac{(n+1)^c - n^c}{(n^c-1)((n+1)^c-1)} = \frac{1}{2^c-1} \quad (c > 1).$$

Putting $x_n = e^{(n+1)t}$, $y_n = e^{-(n+1)t}$ in (13) gives

(21) $$\sum_{n=1}^{\infty} \frac{1}{\cosh(2n+1)t - \cosh t} = \frac{e^{-|t|}}{2\sinh^2 t}\,.$$

Historical note. The formula

(22) $$\sqrt{3} = 2 - \frac{1}{4} - \frac{1}{4.14} - \frac{1}{4.14.194} - \cdots,$$

where $14 = 4^2 - 2$, $194 = 14^2 - 2, \cdots$, was drawn to the attention of I. J. Good by Dr. G. L. Camm in November 1947. (The sequence 4, 14, 194, ⋯ occurs also in tests for primality of the Mersenne numbers [4], p. 235.) The similar formula

(23) $$\sqrt{r} = \frac{(r-1)\alpha_n}{4\beta_{n-1}} - \frac{r-1}{2}\left(\frac{1}{\beta_n} + \frac{1}{\beta_{n+1}} + \cdots\right),$$

where

$$r > 1, \quad \alpha_1 = 2\,\frac{m+1}{m-1}, \quad \alpha_{n+1} = \alpha_n^2 - 2, \quad \beta_0 = 1, \quad \beta_n = \alpha_1\alpha_2\cdots\alpha_n$$

was given in [3]; and formula (8) in [2] and [6]. Hoggatt [5] then noticed that

$$\sum_n \frac{1}{F_{k2^n}}$$

could similarly be summed. All these results follow from deMorgan's formula. I. J. Good noticed the generalization (4) in November 1947, but at that time did not see its application to the Fibonacci and similar sequences and therefore withheld its publication. P. S. Bruckman independently, and recently, noticed the more general formula (14). Alternate methods of proof appear in [7].

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ON THE HARRIS MODIFICATION OF THE EUCLIDEAN ALGORITHM

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V. C. Harris[1] (see D. E. Knuth[2] also) modified the Euclidean algorithm (= algorithm by greatest integers) for finding the gcd of two odd integers $a > b > 1$. The conditions $a = bq + r$, $|r| < b$, $2|r$ define the integers q,r uniquely. In case $r = 0$, stop. In case $r \neq 0$, divide r by its highest power of 2 and obtain c (say); proceed with b, $|c|$ instead of a,b. Denote by $H(a,b)$ the number of steps in this Harris algorithm.

Example: $83 = 47\cdot 1 + 4\cdot 9,\quad 47 = 9\cdot 5 + 2\cdot 1,\quad 9 = 1\cdot 9;\quad H(83,47) = 3.$

Denote by $E(a,b)$ resp. $N(a,b)$ the number of steps in the algorithm by greatest resp. nearest integers for $a > b > 0$. According to Kronecker, $N(a,b) \leqslant E(a,b)$ always. In this note we prove that $H(a,b)$ is sometimes much larger than $E(a,b)$ and sometimes much smaller than $N(a,b)$.

Let

$$c_0 := 1, \quad c_{n+1} = 2c_n + 5 \quad (n \geqslant 0);$$

obviously

$$E(c_{n+1}, c_n) \leqslant 5 \quad (n \geqslant 0).$$

Furthermore, since

$$c_{n+2} = 3c_{n+1} - 2c_n, \quad 2 \nmid c_n \quad (n \geqslant 0),$$

the choice $a_k = c_k$, $b_k = c_{k-1}$ $(k > 0)$ gives

Theorem 1. For every integer $k > 0$ there exist odd integers $a_k > b_k > 0$ with

$$E(a_k, b_k) \leqslant 5, \quad H(a_k, b_k) = k.$$

Let

$$v_0 := 0, \quad v_1 := 1, \quad v_n := 2v_{n-1} + v_{n-2} \quad (n > 1);$$

then

$$(v_{n+1}, v_n) = 1, \quad v_n \leqslant 3^{n-1}, \quad 2|v_n \Leftrightarrow 2|n \quad (n \geqslant 0).$$

[1] *The Fibonacci Quarterly,* Vol. 8, No. 1 (February, 1970), pp. 102–103.
[2] *The Art of Computer Programming,* Vol. 2, "Seminumerical Algorithms," Addison-Wesley Pub., 1969, pp. 300, 316

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