A PRIMER FOR THE FIBONACCI NUMBERS, PART XV
VARIATIONS ON SUMMING A SERIES
OF RECIPROCALS OF FIBONACCI NUMBERS

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It is not easy, in general, to derive the sum of a series whose terms are reciprocals of Fibonacci numbers such that the subscripts are terms of geometric progressions. However, in [1] Good shows that

(1) $$\sum_{n=0}^{\infty} \frac{1}{F_{2^n}} = \frac{7-\sqrt{5}}{2}$$

a problem proposed by Millin [2]. This particular series can be summed in several different ways.

Method I. Write out the first few terms of (1),

$$1, \quad 1+1, \quad 1+1+\frac{1}{3} = \frac{7}{3}, \quad 1+1+\frac{1}{3}+\frac{1}{21} = \frac{50}{21}, \quad \cdots .$$

Now,

$$\frac{50}{21} = 1+\frac{29}{21} = 1+\frac{L_7}{F_8},$$

which suggests that

(2) $$\frac{1}{F_1}+\frac{1}{F_2}+\frac{1}{F_4}+\cdots+\frac{1}{F_{2^n}} = 1+\frac{L_{2^n-1}}{F_{2^n}}.$$

From [3], we write

(3) $$L_m L_{m+1} - L_{2m+1} = (-1)^m$$

from which it follows that

$$1+\frac{L_{2^n-1}}{F_{2^n}}\cdot\frac{L_{2^n}}{L_{2^n}}+\frac{1}{F_{2^{n+1}}} = 1+\frac{L_{2^{n+1}-1}}{F_{2^{n+1}}}$$

since $F_m L_m = F_{2m}$. Thus, we can prove (2) by mathematical induction. If we compute the limit as $n \to \infty$ for (2), then we have the infinite sum of (1), for (see [3])

$$\lim_{n\to\infty}\left(1+\frac{L_{2^n-1}}{F_{2^n}}\right) = \lim_{n\to\infty}\left(1+\frac{L_{2^n}}{F_{2^n}}\cdot\frac{L_{2^n-1}}{L_{2^n}}\right) = 1+\sqrt{5}\cdot\frac{1}{\alpha},$$

where $\alpha = (1+\sqrt{5})/2$, which simplifies to $(7-\sqrt{5})/2$.

The limits used above can be easily derived from the well-known

$$F_n = \frac{\alpha^n-\beta^n}{\alpha-\beta}, \qquad L_n = \alpha^n+\beta^n,$$

where $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$ are the roots of $x^2-x-1=0$.

$$\lim_{n\to\infty}\frac{L_n}{F_n} = \lim_{n\to\infty}(\alpha-\beta)\frac{\alpha^n+\beta^n}{\alpha^n-\beta^n} = \lim_{n\to\infty}(\alpha-\beta)\frac{1+(\beta/\alpha)^n}{1-(\beta/\alpha)^n} = \sqrt{5}$$

since $(\alpha-\beta) = \sqrt{5}$ and $\beta/\alpha < 1$. In an entirely similar manner, we could show that

$$\lim_{n\to\infty} L_{n+r}/L_n = \alpha^r, \qquad \lim_{n\to\infty} F_{n+r}/F_n = \alpha^r .$$

Method II. Returning to the first few terms of (1),

$$\frac{50}{21} = 2 + \frac{8}{21} = 2 + \frac{F_6}{F_8},$$

which suggests

(4)
$$\frac{1}{F_1} + \frac{1}{F_2} + \cdots + \frac{1}{F_{2^n}} = 2 + \frac{F_{2^n-2}}{F_{2^n}}.$$

If we take the limit as $n \to \infty$ of the right-hand side of (4), we obtain $2 + 1/\alpha^2 = (7 - \sqrt{5})/2$. We can prove (4) by induction, since

$$2 + \frac{F_{2^n-2}}{F_{2^n}} + \frac{1}{F_{2^{n+1}}} = 2 + \frac{(F_{2^{n+1}})(F_{2^n-2})/F_{2^n} + 1}{F_{2^{n+1}}} = 2 + \frac{L_{2^n}F_{2^n-2} + 1}{F_{2^{n+1}}}.$$

We need to establish that

$$F_{2^n-2}L_{2^n} + 1 = F_{2^{n+1}-2}$$

which follows from (see [3], [4])

(5)
$$F_{m+p} - F_{m-p} = F_p L_m, \quad p \text{ even},$$

where $m + p = 2^{n+1} - 2,\ m - p = 2,\ m = 2^n,\ p = 2^n - 2$, so that

$$F_{2^{n+1}-2} - F_2 = F_{2^n-2}L_{2^n}.$$

Method III. Examining the first terms of (1) yet again,

$$\frac{50}{21} = 3 - \frac{13}{21} = 3 - \frac{F_7}{F_8}$$

suggests

(6)
$$\frac{1}{F_1} + \frac{1}{F_2} + \cdots + \frac{1}{F_{2^n}} = 3 - \frac{F_{2^n-1}}{F_{2^n}},$$

used by Good [1], where the limit as $n \to \infty$ of the right-hand side is $3 - 1/\alpha = (7 - \sqrt{5})/2$. Establishing (6) by induction involves showing that

$$3 - \frac{F_{2^n-1}}{F_{2^n}} + \frac{1}{F_{2^{n+1}}} = 3 - \frac{L_{2^n}F_{2^n-1} - 1}{F_{2^{n+1}}} = 3 - \frac{F_{2^{n+1}-1}}{F_{2^n}},$$

where we need

$$L_{2^n}F_{2^n-1} = F_{2^{n+1}-1} + F_1$$

which follows from [3], [4]

$$F_{m+p} + F_{m-p} = L_m F_p, \quad p \text{ odd},$$

where $m + p = 2^{n+1} - 1,\ m - p = 1,\ m = 2^n,\ p = 2^n - 1$.

Method IV. Proceeding in a similar manner, we notice that

$$\frac{50}{21} = 4 - \frac{34}{21} = 4 - \frac{F_9}{F_8}$$

and

$$\lim_{n \to \infty}\left(4 - \frac{F_{2^n+1}}{F_{2^n}}\right) = 4 - \alpha = 4 - \frac{1+\sqrt{5}}{2} = \frac{7-\sqrt{5}}{2},$$

if indeed

(7)
$$\frac{1}{F_1} + \frac{1}{F_2} + \cdots + \frac{1}{F_{2^n}} = 4 - \frac{F_{2^n+1}}{F_{2^n}}.$$

Thus one expects

$$L_{2^n}F_{2^n+1} - 1 = F_{2^{n+1}+1}$$

which follows from [3], [4]

(8) $$F_{m+p} + F_{m-p} = L_pF_m, \quad p \text{ even},$$

where $m + p = 2^{n+1} + 1,\ m - p = 1,\ m = 2^n + 1,\ p = 2^n$.

Method V. Again looking at the early terms of (1),

$$\frac{50}{21} = 5 - \frac{F_{10}}{F_8}$$

suggests

(9) $$\frac{1}{F_1} + \frac{1}{F_2} + \cdots + \frac{1}{F_{2^n}} = 5 - \frac{F_{2^n-2}}{F_{2^n}},$$

where the limit of the right-hand side as $n \to \infty$ is $5 - \alpha^2 = 5 - (\alpha + 1) = 4 - \alpha$ again. From the form of (9) and earlier experience, one expects

$$F_{2^n+2}L_{2^n} - 1 = F_{2^{n+1}+2}$$

which follows from (8), where $m + p = 2^{n+1} + 2,\ m - p = 2,\ m = 2^n + 2$ and $p = 2^n$.

Method VI. One last time, we inspect the early terms of (1) to observe

$$\frac{50}{21} = 6 - \frac{76}{21} = 6 - \frac{L_9}{F_8}$$

which has the form of

(10) $$\frac{1}{F_1} + \frac{1}{F_2} + \cdots + \frac{1}{F_{2^n}} = 6 - \frac{L_{2^n+1}}{F_{2^n}}.$$

The proof of (10) by induction depends upon the identity

$$L_{2^n+1}L_{2^n} - 1 = L_{2^{n+1}+1}$$

which follows readily from (3). The limit as $n \to \infty$ of the right-hand side of (10) follows from

$$\lim_{n\to\infty} \frac{L_{2^n+1}}{F_{2^n}} = \lim_{n\to\infty} \frac{L_{2^n}}{F_{2^n}} \cdot \frac{L_{2^n+1}}{L_{2^n}} = \sqrt{5}\cdot\alpha,$$

becoming $6 - \sqrt{5}\cdot\alpha$, which simplifies to $(7 - \sqrt{5})/2$.

Method VII. We again return to the early terms of (1), but we proceed in a different manner.

$$2 + \frac{1}{3} + \frac{1}{21} = 2 + \frac{7+1}{21} = 2 + \frac{L_4 + 1}{F_8}$$

$$2 + \frac{L_4+1}{F_8} + \frac{1}{F_{16}} = 2 + \frac{L_8L_4 + L_8 + 1}{F_{16}} = 2 + \frac{L_{12} + L_8 + L_4 + 1}{F_{16}}.$$

Assume that

(11) $$\sum_{j=0}^{n} 1/F_{2^j} = 2 + \frac{L_{2^n-4} + L_{2^n-8} + L_{2^n-12} + \cdots + L_4 + 1}{F_{2^n}}.$$

Since

$$\lim_{n\to\infty} \frac{L_{m-r}}{F_m} = \sqrt{5}\cdot\alpha^{-r},$$

the limit as $n \to \infty$ of the right-hand side of (11) becomes

$$2 + \sqrt{5}(\alpha^{-4} + \alpha^{-8} + \alpha^{-12} + \cdots) + 0 = 2 + \sqrt{5}\cdot\alpha^{-4}[1/(1-\alpha^{-4})] = 2 + \sqrt{5}[1/(\alpha^4 - 1)]$$
$$= 2 + \sqrt{5}[1/(\alpha^2+1)(\alpha^2-1)] = 2 + \sqrt{5}[1/(\sqrt{5}\alpha)(\alpha)] = 2 + 1/\alpha^2$$

since

$$\alpha^2 = \alpha + 1 \quad \text{and} \quad \alpha^2 + 1 = \alpha + 2 = \frac{1+\sqrt{5}}{2} + 2 = \frac{5+\sqrt{5}}{2} = \sqrt{5}\cdot\alpha.$$

Also, since $\alpha^n = (L_n + F_n\sqrt{5})/2$, $\alpha^2 = (3+\sqrt{5})/2$, and the above becomes

$$2 + 1/\alpha^2 = 2 + (3-\sqrt{5})/2 = (7-\sqrt{5})/2.$$

Here, (11) can be proved by induction if the identity

$$\text{(12)} \qquad L_{2^n}(L_{(2^n-4)} + L_{(2^n-8)} + \cdots + L_4 + 1) = L_{2^{n+1}-4} + L_{2^{n+1}-8} + \cdots + L_4$$

is known. (See [5]).

We could also have used

$$\text{(13)} \qquad \sum_{j=1}^{n} L_{2kj} = \frac{L_{2k(n+1)} - L_{2kn} - L_{2k} - 2}{L_{2k} - 2}$$

to sum the numerator of (11), and proceeded as in [6].

Method VIII. Starting with the first few partial sums,

$$\frac{1}{F_1} + \frac{1}{F_2} = 1 + \frac{L_2}{F_4}, \quad \frac{1}{F_1} + \frac{1}{F_2} + \frac{1}{F_4} = 1 + \frac{L_2+1}{F_4}, \quad \frac{1}{F_1} + \frac{1}{F_2} + \frac{1}{F_4} + \frac{1}{F_8} = 1 + \frac{L_6+L_4+L_2+1}{F_8}$$

Generally,

$$\text{(14)} \qquad \sum_{j=0}^{n} 1/F_{2^j} = 1 + \frac{L_{2^n-2} + L_{2^n-4} + \cdots + L_2 + 1}{F_{2^n}},$$

but

$$L_{2m} + L_{2m-2} + \cdots + L_2 = L_{2m+1} - 1.$$

Thus

$$\text{(15)} \qquad \sum_{j=0}^{n} 1/F_{2^j} = 1 + \frac{L_{2^n-1}}{F_{2^n}} = A$$

so that

$$\lim_{n\to\infty} A = 1 + \sqrt{5}/\alpha = (7-\sqrt{5})/2.$$

Method IX. I. J. Good [7] uses the identity

$$\sum_{n=1}^{\infty} (xy)^{2^{n-1}}/(x^{2^n} - y^{2^n}) = \frac{\text{min abs}(x,y)}{x-y},$$

where $x = (1+\sqrt{5})/2$ and $y = (1-\sqrt{5})/2$. This is not quite complete by itself.

Method X. On the other hand, L. Carlitz [8] uses

$$\sum_{n=0}^{\infty} 1/F_{2^n} = \sum_{i=0}^{\infty} \frac{\alpha-\beta}{\alpha^{2^i} - \beta^{2^i}} = 1 + \sum_{i=1}^{\infty} \frac{\alpha-\beta}{\alpha^{2^i} - \beta^{2^i}} = (\alpha-\beta)\sum_{i=1}^{\infty}\left(\sum_{j=0}^{\infty} \beta^{j2^i}/\alpha^{(j+1)2^i}\right) + 1,$$

but $(\alpha\beta)^2 = 1$, so that this is

$$(\alpha-\beta)\sum_{i=1}^{\infty}\left(\sum_{j=0}^{\infty} \alpha^{-(2j+1)2^i}\right) + 1$$

but clearly, every even number greater than zero can be written as $(2j+1)2^i$. Thus, this is

$$1 + (\alpha-\beta)\sum_{n=1}^{\infty} \alpha^{-2n} = 1 + \frac{\alpha^{-2}(\alpha-\beta)}{1-\alpha^{-2}} = 1 + \frac{\alpha-\beta}{\alpha^2-1} = 1 + \sqrt{5}/\alpha = \frac{7-\sqrt{5}}{2}.$$

Method XI. For yet another method see A. G. Shannon's solution in the April 1976 Advanced Problem Section solution to H-237.

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[Continued from Page 253.]

Then the sequence

$$(w_n) = (\log H_n H_n^*)$$

is u.d. mod 1.

Proof. We have

$$w_{n+1} - w_n = \log \frac{H_{n+1}}{H_n} + \log \frac{H_{n+1}^*}{H_n^*} ,$$

which tends to

$$2 \log \frac{1+\sqrt{5}}{2}$$

as $n \to \infty$ for

$$\frac{H_{n+1}}{H_n} = \frac{qF_n + pF_{n-1}}{qF_{n-1} + pF_{n-2}} = \frac{q(F_n/F_{n-1}) + p}{q(F_{n-1}/F_{n-2}) + p} \cdot \frac{F_{n-1}}{F_{n-2}}$$

goes to

$$\frac{1+\sqrt{5}}{2}$$

as $n \to \infty$

Theorem 3. Let p, q, p^*, q^*, H_n and H_n^* have the same meaning as in Theorem 2. Then the sequence

$$(x_n) = (\log (H_n + H_n^*))$$

is u.d. mod 1.

Proof. By the definitions of H_n and H_n^* we have

$$H_n + H_n^* = (q+q^*)F_{n-1} + (p+p^*)F_{n-2} \quad (n \geqslant 3)$$

and so we see that

$$x_{n+1} - x_n = \log ((H_{n+1} + H_{n+1}^*)/(H_n + H_n^*)) = \log \frac{(q+q^*)F_n + (p+p^*)F_{n-1}}{(q+q^*)F_{n-1} + (p+p^*)F_{n-2}} ,$$

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