DIVISIBILITY PROPERTIES OF RECURRENT SEQUENCES

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1. INTRODUCTION

The Fibonacci numbers, the Fibonacci polynomials, and the generalized Fibonacci polynomials, these latter defined by

$$u_n(x,y) = xu_{n-1}(x,y) + yu_{n-2}(x,y); \qquad u_0(x,y) = 0, \quad u_1(x,y) = 1,$$

all have the following divisibility property:

(1) If $m \mid n$, then $u_m \mid u_n$.

In their recent paper [3], Hoggatt and Long prove (1) and more:

(2) If $m \geq 1$ and $n \geq 1$, then $(u_m, u_n) = u_{(m,n)}$.

Further, if p is a prime, then $u_p(x,y)$ is irreducible over the rational number field, a result originating with Webb and Parberry [5]. Similar results for Lucas polynomials and generalized Lucas polynomials are proved by Bergum and Hoggatt [2].

In this present paper, we consider divisibility properties of certain polynomials which include the generalized Fibonacci polynomials and a modification of the generalized Lucas polynomials as special cases.

Let x, y, z be indeterminants and let

(3) $$f_n = (x+y)f_{n-1} - xyf_{n-2}; \qquad f_0 = 0, \quad f_1 = 1.$$

Define $\ell_0 = 0$, $\ell_1(f) = f = f_1$, and

$$\ell_n(x,y,z) = \ell_n(f) = \begin{cases} f\ell_{n-1}(f) + z\ell_{n-2}(f) & \text{for even } n \\ f\ell_{n-1}(f) + z\ell_{n-2}(f) + 2z^{(n-1)/2} & \text{for odd } n, \end{cases}$$

where f^i is replaced by f_i for $i \geq 0$ after the multiplications involving f are carried out. Since

$$f_n(x,y) = (x^n - y^n)/(x-y)$$

for $n \geq 1$, it is easy to write out the first few $\ell_n(x,y,z)$ as follows:

$$\begin{aligned}
\ell_0 &= 0 = f_0 \\
\ell_1 &= (x-y)/(x-y) = f_1 \\
\ell_2 &= (x^2 - y^2)(x-y) = f_2 \\
\ell_3 &= [x^3 - y^3 + 3z(x-y)]/(x-y) = f_3 + 3zf_1 \\
\ell_4 &= [x^4 - y^4 + 4z(x^2 - y^2)]/(x-y) = f_4 + 4zf_2 \\
\ell_5 &= f_5 + 5zf_3 + 5z^2f_1 \\
\ell_6 &= f_6 + 6zf_4 + 9z^2f_2 \\
\ell_7 &= f_7 + 7zf_5 + 14z^2f_3 + 7z^3f_1 \\
\ell_8 &= f_8 + 8zf_6 + 20z^2f_4 + 16z^3f_2 \\
\ell_9 &= f_9 + 9zf_7 + 27z^2f_5 + 30z^3f_3 + 9z^4f_1
\end{aligned}$$

In general,

(4) $$\ell_n(x,y,z) = \sum_{i=0}^{w} \left[\binom{n+1-i}{i} - \binom{n-1-i}{i-2}\right] z^i f_{n-2i}, \quad \text{where } w = \begin{cases} (n-2)/2 & \text{for even } n \\ (n-1)/2 & \text{for odd } n \end{cases}.$$

*Supported by University of Evansville Faculty Research Grant A6D-0240.

Several special cases of the polynomials $\ell_n(x,y,z)$ are as follows:

Fibonacci numbers	$\ell_n(\alpha,\beta,0)$;	$\alpha+\beta = 1, \quad \alpha\beta = -1$
Fibonacci polynomials	$\ell_n(a,b,0)$;	$a+b = x, \quad ab = -1$
generalized Fibonacci polynomials	$\ell_n(A,B,0)$;	$A+B = x, \quad AB = -y$
modified Lucas numbers	$\ell_n(1,0,1)$	
modified Lucas polynomials	$\ell_n(x,0,1)$	
generalized modified Lucas polynomials	$\ell_n(x,0,z)$	

For comparison with (unmodified) sequences of generalized Lucas polynomials $L_n(x,z)$, Lucas polynomials $L_n(x,1)$, and Lucas numbers $L_n(1,1)$, we have, for $n = 0, 1, \cdots$,

$$L_n(1,1)\text{: } 2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, \cdots,$$

$$\ell_n(1,0,1)\text{: } 0, 1, 1, 4, 5, 11, 16, 29, 45, 76, 121, 199, 320, 521, \cdots;$$

$$L_n(x,z) = xL_{n-1}(x,z) + zL_{n-2}(x,z), \quad L_0(x,z) = 2, \quad L_1(x,z) = x;$$

(5)
$$x\ell_n(x,0,z) = \begin{cases} L_n(x,z) & \text{for odd } n \\ L_n(x,z) - 2z^{n/2} & \text{for even } n; \end{cases}$$

(6)
$$\ell_n(x,y,z) = \frac{L_n(x,z) - L_n(y,z)}{x-y}.$$

In Section 2 we prove that the divisibilities in (1) hold for the polynomials $\ell_n(x,y,z)$. In Section 3 we prove that consecutive terms of the sequence $\ell_n(x,y,z)$ are relatively prime. In Section 4 we prove the same for sequences of the form ℓ_{mn}/ℓ_m, where m is fixed. In Section 5 we prove that (2) holds for the sequence $\ell_n(x,y,z)$. In Section 6 we consider the irreducibility of some of the $\ell_n(x,y,z)$.

2. A MULTISECTION THEOREM

Lemma 1. The sequence $\ell_n(x,y,z)$, for $n = 1, 2, \cdots$, is generated by the function

$$G(x,y,z,t) = \frac{1+zt^2}{(1-xt-zt^2)(1-yt-zt^2)}.$$

Proof. Let

$$x(t) = 1-xt-zt^2 = (1-t_1t)(1-t_2t) \quad \text{and} \quad y(t) = 1-yt-zt^2 = (1-t_3t)(1-t_4t).$$

It is easy to check that

$$(x-y)G(x,y,z,t) = \frac{-x'(t)}{x(t)} - \frac{-y'(t)}{y(t)},$$

and it is well known [1] that $-x'(t)/x(t)$ generates a sequence of sums of powers of roots of $x(t)$. Explicitly,

(7)
$$(x-y)G(x,y,z,t) = \left\{ s_1(x) + s_2(x)t + \cdots - [s_1(y) + s_2(y)t + \cdots] \right\},$$

where $s_n(x) = t_1^n + t_2^n$ is the n^{th} (unmodified) generalized Lucas polynomial $L_n(x,z)$. Thus, the sequence $s_n(x) - s_n(y)$ generated in (7) is $L_n(x,z) - L_n(y,z)$. By (6), the proof is finished.

Theorem 1. If $m \mid n$, where $m \geq 1$ and $n \geq 1$, then $\ell_m(x,y,z) \mid \ell_n(x,y,z)$.

Proof. (The multisection procedure used here is explained in Chapter 4 of Riordan [4].) The $m-1, m$ section of the series in (7) is

$$s_m(x)t^{m-1} + s_{2m}(x)t^{2m-1} + \cdots - [s_m(y)t^{m-1} + s_{2m}(y)t^{2m-1} + \cdots],$$

which we write as

$$(x-y)(\ell_m t^{m-1} + \ell_{2m} t^{2m-1} + \cdots).$$

Again as in [1], we know that $s_m(x) + s_{2m}(x)t + \cdots = -X'(t)/X(t)$, where

$$X(t) = (1-t_1^m t)(1-t_2^m t) = 1 - s_m(x)t + (-z)^m t^2,$$

and similarly,

$$s_m(y) + s_{2m}(y)t + \cdots \quad -Y'(t)/Y(t),$$

where

$$Y(t) = (1 - t_3^m t)(1 - t_4^m t) = 1 - s_m(y)t + (-z)^m t^2.$$

Thus,

$$(x - y)(\ell_m + \ell_{2m}t + \cdots) = \frac{-X'(t)}{X(t)} - \frac{-Y'(t)}{Y(t)} = \frac{XY' - X'Y}{XY}.$$

Write $1/XY$ as $H_0 + H_1 t + H_2 t^2 + \cdots$. Then

$$\ell_m + \ell_{2m}t + \cdots = \frac{1}{x - y}(XY' - X'Y)(H_0 + H_1 t + H_2 t^2 + \cdots)$$
$$= \ell_m[1 - (-z)^m t^2](H_0 + H_1 t + H_2 t^2 + \cdots).$$

Therefore, if $n = km$, then $\ell_n = \ell_m[H_{k-1} - (-z)^m H_{k-3}]$.

3. CONSECUTIVE RELATIVELY PRIME POLYNOMIALS

The $m - 1$, m section of $xG(x,0,z,t) = L_1 + L_2 t + L_3 t^2 + \cdots$ readily provides a well known (e.g., [4]) recurrence relation

$$L_{nm} = L_m L_{(n-1)m} - (-z)^m L_{(n-2)m}$$

for subsequences of (unmodified) generalized Lucas polynomials. Substituting for the L's according to (5), we readily obtain the following lemma for modified generalized Lucas polynomials.

Lemma 2. Let $\ell_n = \ell_n(x,0,z)$ for $n \geq 0$. Then for $n \geq 2$,

$$\ell_{nm} = \begin{cases} \ell_m(x\ell_{(n-1)m} + 2z^{(n-1)m/2}) + z^m \ell_{(n-2)m} & \text{for odd } m \text{ and odd } n \\ x\ell_m \ell_{(n-1)m} + z^m \ell_{(n-2)m} & \text{for odd } m \text{ and even } n; \end{cases}$$

$$\ell_{nm} = (x\ell_m + 2z^{m/2})\ell_{(n-1)m} - z^m \ell_{(n-2)m} + 2z^{(n-1)m/2}\ell_m \quad \text{for even } m.$$

Theorem 2. Let $\ell_n = \ell_n(x,0,z)$ for $n \geq 0$. Then for $n \geq 1$,

$$xz^{n-1} = \begin{cases} -x(\ell_n + 2z^{(n-1)/2})\ell_n + (x\ell_{n-1} + 4z^{(n-1)/2})\ell_{n+1} & \text{for odd } n \\ (x\ell_n + 4z^{n-2})\ell_n - x\ell_{n-1}\ell_{n+1} & \text{for even } n. \end{cases}$$

Proof. The proposition is obviously valid for $n = 1$. Suppose its validity for arbitrary odd n. Then for any even n,

$$\begin{aligned} xz^{n-2} &= (x\ell_{n-2} + 4z^{(n-2)/2})\ell_n - x(\ell_{n-1} + 2z^{(n-2)/2})\ell_{n-1} \\ xz^{n-1} &= (xz\ell_{n-2} + 4z^{n/2})\ell_n - x(z\ell_{n-1} + 2z^{n-2})\ell_{n-1} \\ &= (x\ell_n - x^2\ell_{n-1} + 4z^{n/2})\ell_n - x\ell_{n-1}(z\ell_{n-1} + 2z^{n/2}) \\ &= (x\ell_n + 4z^{n/2})\ell_n - x\ell_{n-1}(x\ell_n + z\ell_{n-1} + 2z^{n/2}) \\ &= (x\ell_n + 4z^{n/2})\ell_n - x\ell_{n-1}\ell_{n+1}. \end{aligned}$$

Now suppose the proposition valid for arbitrary even n. Then for any odd n,

$$\begin{aligned} xz^{n-2} &= -x\ell_{n-2}\ell_n + (x\ell_{n-1} + 4z^{(n-2)/2})\ell_{n-1} \\ xz^{n-1} &= -xz\ell_{n-2}\ell_n + (x\ell_{n-1} + 4z^{(n-1)/2})z\ell_{n-1} \\ &= -x(\ell_n - x\ell_{n-1} - 2z^{(n-1)/2})\ell_n + (x\ell_{n-1} + 4z^{(n-1)/2})z\ell_{n-1} \\ &= -x(\ell_n - 2z^{(n-1)/2})\ell_n + (x\ell_{n-1} + 4z^{(n-1)/2})z\ell_{n-1} + x^2\ell_n\ell_{n-1} \\ &= -x(\ell_n + 2z^{(n-1)/2})\ell_n + (x\ell_{n-1} + 4z^{(n-1)/2})z\ell_{n-1} + x^2\ell_n\ell_{n-1} + 4xz^{(n-1)/2}\ell_n \\ &= -x(\ell_n + 2z^{(n-1)/2})\ell_n + (x\ell_{n-1} + 4z^{(n-1)/2})(z\ell_{n-1} + x\ell_n) \\ &= -x(\ell_n + 2z^{(n-1)/2})\ell_n + (x\ell_{n-1} + 4z^{(n-1)/2})\ell_{n+1}. \end{aligned}$$

Corollary 2. Let $\ell_n = \ell_n(x,0,z)$ for $n \geq 0$. Then $(\ell_n, \ell_{n+1}) = 1$ for $n \geq 1$.

Proof. Theorem 2 shows that the only possible divisors of both ℓ_n and ℓ_{n+1} are of the form $x^i z^j$ where $0 \leq i \leq 1$ and $0 \leq j \leq n - 1$. Equation (4) shows that $j = 0$. Reading x for f in (4), we see that x divides $\ell_n(x,0,z)$ only when n is even. Since $x^i z^0$ divides consecutive ℓ's, we have $i = 0$.

Lemma 3. Let $\ell_n = \ell_n(x,y,0) = f_n$ for $n \geq 0$. Then $(\ell_n, \ell_{n+1}) = 1$ for $n \geq 1$.

Proof. Clearly $(f_1,f_2) = (f_2,f_3) = 1$. Suppose for arbitrary $n \geq 2$ that $(f_{n-2}, f_{n-1}) = 1$. If $d(x,y)$ divides both f_{n-1} and f_n then by (3), $d(x,y)$ divides xyf_{n-2}. Thus $d(x,y)$ divides f_{n-2}, since $(xy,f_{n-2}) = 1$. But the only divisor of both f_{n-2} and f_{n-1} is 1. Therefore $d(x,y) = 1$.

Theorem 3. Let $\ell_n = \ell_n(x,y,z)$ for $n \geq 0$. Then $(\ell_n, \ell_{n+1}) = 1$ for $n \geq 1$.

Proof. Suppose $d(x,y,z)$ divides both $\ell_n(x,y,z)$ and $\ell_{n+1}(x,y,z)$. Then $d(x,y,0)$ divides both $\ell_n(x,y,0)$ and $\ell_{n+1}(x,y,0)$. By Lemma 3, $d(x,y,z) = 1 + ze(x,y,z)$ for some $e(x,y,z)$. Since $1 + ze(x,y,z)$ divides $\ell_n(x,y,z)$, we have

$$\ell_n(x,y,z) = q(x,y,z) + ze(x,y,z)q(x,y,z)$$

for some $q(x,y,z)$. Now z does not divide $q(x,y,z)$, since z does not divide $\ell_n(x,y,z)$. Therefore the term x^{n-1} in $\ell_n(x,y,z)$ occurs in $q(x,y,z)$, Consequently, unless $e(x,y,z)$ is the zero polynomial, some nonzero multiple of zx^{n-1} occurs in the polynomial $ze(x,y,z)q(x,y,z)$. But $\ell_n(x,y,z)$ has no such term. Therefore $e(x,y,z)$ is the zero polynomial, so that $d(x,y,z) = 1$.

4. SUBSEQUENCES OF ℓ_n

In this section we consider subsequences of the form $\ell_m, \ell_{2m}, \ell_{3m}, \cdots$, where $m \geq 1$. Since each term is divisible by the first term, let us divide all terms by the first, and let $\lambda_{m,n} = \ell_{nm}/\ell_m$ for $n \geq 0$. Then by Theorem 1, for $m \geq 1$ and $k \geq 1$, $\lambda_{m,k} | \lambda_{m,n}$ whenever $k|n$. Do the λ sequences also inherit from the ℓ sequence the property that consecutive terms are relatively prime?

Lemma 4a. Let $\lambda_n = \lambda_{m,n}(x,0,z)$ for $n \geq 0$. Then for $n \geq 1$,

$$x\ell_{(n-1)m} = \begin{cases} \lambda_2\lambda_{n-1} & \text{for odd } m \\ (\lambda_2 - 4z^{m/2})\lambda_{n-1} & \text{for even } m. \end{cases}$$

Proof. By Lemma 2,

$$\ell_{2m} = \begin{cases} x\ell_m^2 & \text{for odd } m \\ x\ell_m^2 + 4z^{m/2}\ell_m & \text{for even } m. \end{cases}$$

Thus, for odd m,

$$x\ell_{(n-1)m} = \frac{\ell_{2m}}{\ell_m^2}\,\ell_{(n-1)m} = \lambda_2\lambda_{n-1}.$$

For even m,

$$x\ell_{(n-1)m} = \frac{(\ell_{2m} - 4z^{m/2}\ell_m)\ell_{(n-1)m}}{\ell_m^2} = (\lambda_2 - 4z^{m/2})\lambda_{n-1}.$$

Lemma 4b. Let $\lambda_n = \lambda_{m,n}(x,0,z)$ for $n \geq 0$. Then for odd m and $n \geq 2$,

$$\lambda_n = \begin{cases} \lambda_2\lambda_{n-1} + z^m\lambda_{n-2} + 2z^{(n-1)m/2} & \text{for odd } n \\ \lambda_2\lambda_{n-1} + z^m\lambda_{n-2} & \text{for even } n; \end{cases}$$

and for even m,

$$\lambda_n = (\lambda_2 - 2z^{m/2})\lambda_{n-1} - z^m\lambda_{n-2} + 2z^{(n-1)m/2},$$

with $\lambda_0 = 0$ and $\lambda_1 = 1$.

Proof. In Lemma 2, divide both sides of the three recurrence relations by ℓ_m, recalling that $\lambda_k = \ell_{km}/\ell_m$ for $k = n, n-1, n-2$. Now replace $x\ell_{(n-1)m}$ by $\lambda_2\lambda_{n-1}$ for odd m, and replace $x\ell_m$ by $\lambda_2 - 4z^{m/2}$ for even m.

Theorem 4a. Let $\lambda_n = \lambda_{m,n}(x,0,z)$ for $n \geq 0$. Then for odd m and $n \geq 1$,

$$\lambda_2 z^{(n-1)m} = \begin{cases} -\lambda_2(\lambda_n + 2z^{(n-1)m/2})\lambda_n + (\lambda_2\lambda_{n-1} + 4z^{(n-1)m/2})\lambda_{n+1} & \text{for odd } n \\ (\lambda_2\lambda_n + 4z^{nm/2})\lambda_n - \lambda_2\lambda_{n-1}\lambda_{n+1} & \text{for even } n. \end{cases}$$

Proof. Referring to the proof of Theorem 1, we know that for odd m,

$$\ell_m + \ell_{2m} + \cdots = \ell_m \frac{1+z^m t^2}{X(t)Y(t)} ,$$

so that

$$\lambda_1 + \lambda_2 t + \cdots = G[s_m(x), s_m(y), z^m, t] .$$

Since $s_m(y) = L_n(y,z)$, as in the proof of Lemma 1, Eqs. (5) and (6) show that $s_m(y) = 0$ for all odd m. Further, $s_m(x) = x\ell_m$, which by Lemma 4a equals ℓ_{2m}/ℓ_m, which is λ_2. Therefore,

$$\lambda_1 + \lambda_2 t + \cdots = G(\lambda_2, 0, z^m, t) = \ell_1(\lambda_2, 0, z^m) + \ell_2(\lambda_2, 0, z^m)t + \cdots,$$

by Lemma 1. Thus Theorem 2 applies with x and z replaced by λ_2 and z^m, respectively. The result is exactly as stated above.

Theorem 4b. Let $\lambda_n = \lambda_{m,n}(x,0,z)$ for $n \geq 0$. Then for even m and $n \geq 1$,

$$z^{(n-1)m} = -(\lambda_n - 2z^{(n-1)m/2})\lambda_n + \lambda_{n-1}\lambda_{n+1} .$$

Proof. The proposition is clearly valid for $n = 1$. For arbitrary $n > 1$, suppose that

$$z^{(n-2)m} = -(\lambda_{n-1} - 2z^{(n-2)m/2})\lambda_{n-1} + \lambda_{n-2}\lambda_n .$$

Then

$$z^{(n-1)m} = -[-\lambda_{n+1} + (\lambda_2 - 2z^{m/2})\lambda_n]\lambda_{n-1} - [\lambda_n - (\lambda_2 - 2z^{m/2})\lambda_{n-1} - 2z^{(n-1)m/2}]\lambda_n$$

by Lemma 4b

$$= -(\lambda_n - 2z^{(n-1)m/2})\lambda_n + \lambda_{n-1}\lambda_{n+1} .$$

Corollary 4. Let $\lambda_n = \lambda_{m,n}(x,0,z)$ for $n \geq 0$. Then $(\lambda_n, \lambda_{n+1}) = 1$ for all positive integers m and n.

Proof. For odd m, Theorem 4a shows that the only possible divisors of both λ_n and λ_{n+1} are of the form $\lambda_2^i z^j$, where $0 \leq i \leq 1$ and $0 \leq j \leq (n-1)m$. As in the proof of Theorem 4a, $\lambda_n = \ell_n(\lambda_2, 0, z^m)$, so that (4) gives (for *odd m* only),

$$\lambda_n = \sum_{i=0}^{w} \left[\binom{n+1-i}{i} - \binom{n-1-i}{i-2}\right] z^{mi} f^*_{n-2i}, \text{ where } w = \begin{cases} (n-2)/2 & \text{for even } n \\ (n-1)/2 & \text{for odd } n, \end{cases}$$

and $f_k^*(x,0) = f_k(\lambda_2,0) = \lambda_2^{k-1}$ for $k \geq 1$. Thus λ_2 divides λ_n only when n is even. Since λ_2^i divides consecutive λ's, we have $i = 0$.

Now for *any m*, Theorem 4b and the argument just given show that the only possible divisors of both λ_n and λ_{n+1} are of the form z^j with $0 \leq j \leq (n-1)m$. If z^j divides λ_n then z^j divides $\ell_{mn} = \ell_m \ell_n$. Thus $j = 0$, by (4).

Lemma 5. Let $\lambda_n = \lambda_{m,n}(x,y,0)$ for $n \geq 0$. Then $(\lambda_n, \lambda_{n+1}) = 1$ for $n \geq 1$.

Proof. Since $z = 0$, we have $\lambda_n = f_{nm}/f_m = f_n(x^m, y^m)$. Now (3) is used to complete the proof, just as in the proof of Lemma 3.

Theorem 5. Let $\lambda_n = \lambda_{m,n}(x,y,z)$ for $n \geq 0$. Then $(\lambda_n, \lambda_{n+1}) = 1$ for $n \geq 1$.

Proof. The method of proof is exactly as for Theorem 3. Here the exponent of x to be considered is $^{m(n-1)}$ rather than $^{n-1}$.

Theorem 6. Let m and n be odd. Let $\lambda_n = \lambda_{m,n}(x,0,z)$ for $n \geq 0$. Then for $n \geq 1$, $\lambda_n \equiv nz^{(n-1)m/2} \bmod \lambda_2$.

Proof. By Lemma 4b,

$$\lambda_n \equiv (z^m \lambda_{n-2} + 2z^{(n-1)m/2}) \bmod \lambda_2 \text{ for odd } n.$$

Repeated application of this congruence gives

$$\lambda_n \equiv 2z^{(n-1)m/2} + z^m\lambda_{n-2} \equiv 2z^{(n-1)m/2} + z^m(z^m\lambda_{n-4} + 2z^{(n-3)m/2}) \equiv 4z^{(n-1)m/2} + z^{2m}\lambda_{n-4} \equiv \cdots$$

$$\equiv 2kz^{(n-1)m/2} + z^{km}\lambda_{n-2k} \text{ for } k = 1, 2, \cdots, \frac{n-1}{2} .$$

In particular, for $k = \frac{n-1}{2}$,

$$\lambda_n \equiv (n-1)z^{(n-1)m/2} + z^{(n-1)m/2}\lambda_1 ,$$

as desired.

Corollary 6a. Let m be an odd positive integer.

Let $\lambda_n = \lambda_{m,n}(1,0,1)$ for $n \geq 0$. Then $\lambda_2 = \ell_m$. If n is odd and $n \equiv 0 \bmod \lambda_m$, then $(\lambda_n, \lambda_{n+1}) = \ell_m$. If ℓ_m is a prime, then $(\lambda_n, \lambda_{n+1}) = 1$ for each positive integer n satisfying $n \not\equiv 0 \bmod \ell_m$. For $m = 1$ and $m = 3$,

$$(\lambda_n, \lambda_{n+1}) = 1 \quad \text{for} \quad n \geq 1.$$

Proof. By Lemma 4a, $\ell_{2m} = \ell_m^2$, so that $\lambda_2 = \ell_m$. If n is odd, then $\lambda_n \equiv n \bmod \lambda_2$, by Theorem 6. Thus $\lambda_n \equiv n \bmod \ell_m$, and if $n \equiv 0 \bmod \ell_m$, then $\lambda_n \equiv 0 \bmod \ell_m$. Since $n + 1$ is even, λ_2 divides λ_{n+1}, by Theorem 1. Therefore both λ_n and λ_{n+1} are divisible by λ_2. By Theorem 4a, the only divisors of both λ_n and λ_{n+1} are divisors of λ_2. Therefore $\lambda_2 = (\lambda_n, \lambda_{n+1})$.

Now for n either odd or even, Theorem 4a still shows that the only divisors of both λ_n and λ_{n+1} are divisors of λ_2. Thus if $\lambda_2 = \ell_m$ is a prime, then the conditions $\lambda_n \equiv n \bmod \ell_m$ and $n \not\equiv 0 \bmod \ell_m$ show that ℓ_m does not divide λ_n. Thus $(\lambda_n, \lambda_{n+1}) = 1$.

For $m = 1$, put $x = z = 1$ in Theorem 2.

For $m = 3$, we have $\lambda_2 = 4$, so that after dividing by 4 in Theorem 4a, we find

$$1 = \begin{cases} -(\lambda_n + 2)\lambda_n + (\lambda_{n-1} + 1)\lambda_{n+1} & \text{for odd } n \\ (\lambda_n + 1)\lambda_n - \lambda_{n-1}\lambda_{n+1} & \text{for even } n. \end{cases}$$

Corollary 6b. Let m be an even positive integer. Let $\lambda_n = \lambda_{m,n}(1,0,1)$ for $n \geq 0$. Then $(\lambda_n, \lambda_{n+1}) = 1$ for $n \geq 1$.

Proof. This is an immediate consequence of Theorem 4b.

Example. To illustrate Corollary 6a, let $m = 5$. Recalling the abbreviation

$$\lambda_5 = \lambda_{5,n}(1,0,1) = \ell_{5n}(1,0,1)/\ell_5(1,0,1)$$

for $n \geq 0$ and Lemma 4b, we have for $n \geq 2$:

$$\lambda_n = \begin{cases} 11\lambda_{n-1} + \lambda_{n-2} + 2 & \text{for odd } n \\ 11\lambda_{n-1} + \lambda_{n-2} & \text{for even } n. \end{cases}$$

We write out the first 12 λ's and factor them

$$\begin{aligned}
\lambda_0 &= 0 = 0 \\
\lambda_1 &= 1 = 1 \\
\lambda_2 &= 11 = 11 \\
\lambda_3 &= 124 = 2^2 \cdot 31 \\
\lambda_4 &= 1375 = \lambda_2(\lambda_2^2 + 4) \\
\lambda_5 &= 15251 = 101 \cdot 151 \\
\lambda_6 &= 169136 = \lambda_2\lambda_3^2 \\
\lambda_7 &= 1875749 = 29 \cdot 71 \cdot 911 \\
\lambda_8 &= 20802375 = \lambda_4(\lambda_2\lambda_4 + 4) \\
\lambda_9 &= 230701876 = \lambda_3(\lambda_2\lambda_6 + 3) \\
\lambda_{10} &= 2558523011 = \lambda_2\lambda_5^2 \\
\lambda_{11} &= 28374454999 = \lambda_2 \cdot 199 \cdot 331 \cdot 39161.
\end{aligned}$$

In agreement with Corollary 6a, we have $(\lambda_n, \lambda_{n+1}) = 1$ for $1 \leq n \leq 9$, but $(\lambda_{10}, \lambda_{11}) = 11$.

5. THE EQUATION $(\ell_m, \ell_n) = \ell_{(m,n)}$

Lemma 7. Let $\lambda_j = \lambda_{m,j}(x,y,z)$ for $m \geq 1$ and $j \geq 0$. If k is a positive integer satisfying $(j,k) = 1$, then $(\lambda_j, \lambda_k) = 1$.

Proof. Write $1 = sj - tk$ (or $sk - tj$) where s and t are nonnegative. Let $d = (\lambda_j, \lambda_k)$. Then $d | \lambda_{sj}$ and $d | \lambda_{tk}$ by Theorem 1. Thus $d | (\lambda_{sj}, \lambda_{tk})$, which is to say $d | (\lambda_{tk}, \lambda_{tk+1})$. By Theorem 5, we have $d = 1$.

Theorem 7. Let $\ell_m = \ell_m(x,y,z)$ for $m \geq 0$. Then for $m \geq 1$ and $n \geq 1$, $(\ell_m, \ell_n) = \ell_{(m,n)}$.

Proof. Let $d = (m,n)$. Let $j = m/d$ and $k = k/d$. Then $(\lambda_j, \lambda_k) = 1$, by Lemma 7. Now $\ell_m = \lambda_j \ell_d$ and $\ell_n = \lambda_k \ell_d$. Therefore $(\ell_m, \ell_n) = \ell_d$.

Lemma 8. The following items hold true when restated in terms of $(x,0,1)$ and $(1,0,1)$, instead of $(x,0,z)$: Lemmas 4a and 4b, Theorems 4a and 4b, Theorem 6, and Lemma 7.

Proof. The resulting restatements are special cases, to which the proofs already given apply.

Theorem 8. The equations $(\ell_m, \ell_n) = \ell_{(m,n)}$ and $(\lambda_j, \lambda_n) = \lambda_{(j,k)}$ as in Theorem 7 and Corollary 7 hold if ℓ and λ are applied to $(x,0,z)$ and $(x,0,1)$. They also hold for $(1,0,1)$ if m is even or equal to 1 or 3.

Proof. The first statement follows from Lemma 8 exactly as Theorem 7 and Corollary 7 follow from Lemma 7.

For the second statement, we obtain $d | (\lambda_{tk}, \lambda_{tk+1})$ as in the proof of Lemma 7 and have $d = 1$ by Corollaries 6a and 6b. Then the methods of proof of Theorem 7 and Corollary 7 apply.

6. IRREDUCIBILITY OF ℓ POLYNOMIALS

Lemma 9. The polynomial $\ell_n(x,y,0)$ is irreducible over the rational number field if and only if n is a prime.

Proof. It is known [3] that the generalized Fibonacci polynomial $\ell_n(A,B,0)$, where $A + B = x$ and $AB = -y$, is irreducible if and only if n is a prime. The present lemma is an immediate consequence.

Theorem 9. The polynomial $\ell_n(x,y,z)$ is irreducible over the rational number field if and only if n is a prime.

Proof. If n is not a prime, we have Theorem 1. Suppose n is a prime and that $\ell_n(x,y,z) = d(x,y,z)q(x,y,z)$. Then one of the polynomials $d(x,y,0)$ and $q(x,y,0)$ must be the constant 1 polynomial, by Lemma 9. Supposing this one to be $d(x,y,0)$, we have $d(x,y,z) = 1 + ze(x,y,z)$ for some $e(x,y,z)$. The remainder of the proof is identical to that of Theorem 3.

Lemma 10. The polynomial $\ell_n(x,0,z) + 2z^{n/2}$, where n is even, is irreducible over the rational number field if and only if $n = 2^k$ for some $k \geq 1$. Further, the polynomial $\ell_n(x,0,z)$ for odd n is irreducible if and only if n is a prime.

Proof. These two results are proved in [2].

Theorem 10. If $n = 2^k$ for some $k \geq 1$, then the polynomial $\ell_n(x,y,z) + 2z^{n/2}$ is irreducible over the rational number field.

Proof. Suppose $\ell_n(x,y,z) + 2z^{n/2} = d(x,y,z)q(x,y,z)$. Then one of the polynomials $d(x,0,z)$ and $q(x,0,z)$ must be the constant 1 polynomial. Supposing this one to be $d(x,0,z)$, we have $d(x,y,z) = 1 + ye(x,y,z)$ for some $e(x,y,z)$, by Lemma 10. Consequently,

$$\ell_n(x,y,z) + 2z^{n/2} = q(x,y,z) + ye(x,y,z)q(x,y,z).$$

Once again, the remainder of the proof is identical to that of Theorem 3, except that here we have yx^{n-1} instead of zx^{n-1}.

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