RESTRICTED COMBINATIONS AND COMPOSITIONS

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INTRODUCTION

The number of k-combinations of $\{1, 2, \cdots, n\}$ with no two consecutive integers in a combination is

$$\binom{n-k+1}{k}$$

while the number of such restricted "circular" k-combinations, that is when 1 and n are also considered as consecutive integers, is

$$\frac{n}{n-k}\binom{n-k}{k} .$$

These are two well known examples of restricted combinations given by Kaplansky [1943] as preliminary problems in his elegant solution of the "problème des ménages." Some other examples are given by Abramson [1971], Church [1966, 1968, 1971] and Moser and Abramson [1969a, b].

In this paper, generating functions and recurrence relations are given for a large class of restricted combinations. This method seems to be a more unified approach than using combinatorial arguments such as those of Moser and Abramson [1969a] whose main result is obtained here in Section 7 as a special case of a more general result.

We take a k-composition of an integer n to be an ordered sequence of non-negative integers $a_1, a_2, \cdots, a_k$, whose sum is n. A one-to-one correspondence between the k-compositions of n with each summand $a_i > 0$ and the $(k-1)$-combinations of $\{1, 2, \cdots, n-1\}$ is obtained by representing the combinations and compositions by binary sequences, see also Abramson and Moser [1976]. Hence there is a correspondence between restricted combinations and restricted compositions. Also, there is a correspondence between "circular" combinations and "circular" compositions.

A k-composition of n may be interpreted of course as an occupancy problem of distributing n like objects in k distinct cells, with a_i objects in cell i. Further a k-composition, $a_1, a_2, \cdots, a_k$ of n corresponds to an n-combintion, with repetitions allowed, from $\{1, 2, \cdots, k\}$ with the integer i appearing a_i times. Also since every binary sequence corresponds to a lattice path we have a 1:1 correspondence between lattice paths in a rectangular array and combinations. For example expression (2.3) of Church [1970] is case (L) of Section 3 here. Some results on combinations which have been obtained by Church and Gould [1967] by counting lattice paths have been generalized by Moser and Abramson [1969 b] and can also be derived using our approach here.

Sections 1 to 5 deal with linear compositions and combinations and Sections 6 and 7 with circular compositions and combinations. Throughout we take, as usual,

$$\binom{n}{k} = \begin{cases} n!/(n-k)!k!, & 0 \leqslant k \leqslant n \\ 0 & \text{otherwise.} \end{cases}$$

1. RESTRICTED COMPOSITIONS

A k-composition of n

$$a_1 + a_2 + \cdots + a_k = n, \qquad a_i \geqslant 1, \tag{1.1}$$

is an ordered sequence of k positive integers a_i, called the summands or parts satisfying (1.1) for fixed n and k. It is well known and easy to show the number of compositions (1.1) is $\binom{n-1}{k-1}$. Let

$$A = (A_1, A_2, \cdots, A_k), \qquad A_i = \left\{ a_{i1} < a_{i2} < a_{i3} < \cdots \right\} \tag{1.2}$$

denote a given collection of k, not necessarily distinct, subsets A_i, of $\{1, 2, 3, \cdots\}$. Denote by $F(n, k; A)$ the number of compositions (1.1) satisfying the restrictions $a_i \in A_i$, $i = 1, 2, \cdots, k$. That is

$$F(n, k; A) = \sum_{\substack{a_1+\cdots+a_k=n \\ a_i \in A_i}} 1 . \tag{1.3}$$

The enumerator generating function as is well known, see Riordan [1958] provides a general method of finding $F(n, k; A)$. This is

$$\sum_n F(n, k; A)x^n = (x^{a_{11}} + x^{a_{12}} + \cdots)(x^{a_{21}} + x^{a_{22}} + \cdots) \cdots (x^{a_{k1}} + x^{a_{k2}} + \cdots) . \tag{1.4}$$

For example, in the case $A_i = \{1, 2, 3, \cdots\}$ for all i

$$\sum_{n=1} F(n, k; A)x^n = (x + x^2 + x^3 + \cdots)^k = \sum_{i=0} \binom{k+i-1}{i} x^{i+k} = \sum_{n=1} \binom{n-1}{k-1} x^n .$$

To each of the compositions (1.1) there corresponds a unique sequence of $n - k$ 0's and $k - 1$ 1's:

$$\underbrace{000\cdots01}_{a_1-1}\quad \underbrace{000\cdots01}_{a_2-1}\quad \cdots \quad \underbrace{000\cdots01}_{a_{k-1}-1}\quad \underbrace{000\cdots0}_{a_k-1} \tag{1.5}$$

Note that since $a_i \geq 1$ in each part of (1.5) the 1 always appears except for the last part where we have a "missing" 1. Replacing the 1's by 0's and 0's by 1's in (1.5) we have a dual representation,

$$\underbrace{111\cdots10}_{a_1-1}\quad \underbrace{111\cdots10}_{a_2-1}\quad \cdots \quad \underbrace{111\cdots10}_{a_{k-1}-1}\quad \underbrace{111\cdots1}_{a_k-1} \tag{1.6}$$

corresponding to a unique sequence of $n - k$ 1's and $k - 1$ 0's.

2. RESTRICTED COMBINATIONS

We call r integers

$$x_1 < x_2 < \cdots < x_r , \tag{2.1}$$

chosen from $\{1, 2, \cdots, m\}$ an r-combination (choice, selection) of n. A *part* of (2.1) is a sequence of consecutive integers not contained in a longer sequence of consecutive integers. In a combination (2.1) a *succession* is a pair x_i, x_{i+1} with $x_{i+1} - x_i = 1$. It is easy to see that if a combination has q parts then it has $r - q$ successions. For example

$$1, 3, 4, 5, 8, 9 \tag{2.2}$$

is a 6-combination of 10, with parts (1), (3, 4, 5), (8, 9) of lengths 1, 3, 2, respectively. To each combination (2.1) corresponds a unique sequence of r 1's and $m - r$ 0's

$$e_1, e_2, e_3, \cdots, e_m , \tag{2.3}$$

where $e_i = \begin{cases} 1 \text{ if } i \text{ belongs to the } r\text{-combination} \\ 0 \text{ if } i \text{ does not belong to the } r\text{-combination.} \end{cases}$

For the combination (2.2) the corresponding sequence is

$$1011100110 . \tag{2.4}$$

To a given restricted composition (1.1) corresponds by the use of (1.5) a unique $(k-1)$-combination

$$x_1 < x_2 < \cdots < x_{k-1} \tag{2.5}$$

of $n - 1$ such that

$$x_1 = a_1, \quad n - x_{k-1} = a_k, \quad x_{i+1} - x_i = a_{i+1}, \quad i = 1, 2, \cdots, k-2. \tag{2.6}$$

Hence $F(n, k; A)$ is the number of combinations (2.5) satisfying the restrictions

$$(2.7)\qquad x_1 \in A_1, \quad n - x_{k-1} \in A_k, \quad x_{i+1} - x_i \in A_i, \quad i = 1, 2, \cdots, k-2.$$

For convenience, letting $n - 1 = m$, $k - 1 = r$, $F(m+1, r+1; A)$ is the number of combinations (2.1) satisfying

$$(2.8)\qquad x_1 \in A_1, \quad n - x_r \in A_{r+1}, \quad x_{i+1} - x_i \in A_i, \quad i = 1, 2, \cdots, r-1,$$

where

$$(2.9)\qquad A = (A_1, A_2, \cdots, A_{r+1}), \quad A_i = \left\{ a_{i1} < a_{i2} < \cdots \right\}, \quad i = 1, \cdots, r+1$$

are the given restrictions.

3. EXAMPLES OF RESTRICTED COMPOSITIONS AND COMBINATIONS

Denote by $F(n, k; h_1, p_1; h_2, p_2; \cdots; h_k, p_k)$ the number of k-compositions of n satisfying the restrictions

$$(3.1)\qquad 1 \leqslant h_i \leqslant a_i \leqslant p_i, \quad \text{for fixed} \quad h_i, p_i, \quad i = 1, \cdots, k.$$

Using the **sieve** formula or the enumerator generating function (1.4) with $A_i = \left\{ h_i, h_i + 1, h_i + 2, \cdots, p_i \right\}$, $i = 1, \cdots, k$,

$$(3.2)\qquad F(n,k; h_1, p_1; \cdots; h_k, p_k) = \binom{n-h+k-1}{k-1}$$

$$+ \sum_{j=1}^{k} (-1)^j \sum{}^{*} \binom{n-h+k-j-1-(p_{i_1}-h_{i_1})-(p_{i_2}-h_{i_2})-\cdots-(p_{i_j}-h_{i_j})}{k-1}$$

with $h = h_1 + \cdots + h_k$ and the summation Σ^* taken over all j-combinations $i_1 < i_2 < \cdots < i_j$ of $\left\{ 1, 2, \cdots, k \right\}$. We consider now some special cases.

(A) The number of compositions (1.1) satisfying $1 \leqslant h_i \leqslant a_i$, $i = 1, \cdots, k$, is the case $p_i = n$, $i = 1, \cdots, k$ of (3.2),

$$F(n, k; h_1, n; \cdots; h_k, n) = \binom{n+k-1-h_1-h_2-\cdots-h_k}{k-1}.$$

(B) The number of compositions (1.1) satisfying $1 \leqslant a_i \leqslant p_i$, is the case $h_i = 1$ for all i, of (3.2) which is

$$F(n, k; 1, p_1; \cdots; 1, p_k) = \binom{n-1}{k-1} + \sum_{j=1}^{k} (-1)^j \sum{}^{*} \binom{n-1-p_{i_1}-p_{i_2}-\cdots-p_{i_j}}{k-1}$$

the summation Σ^* taken over all j-combinations $i_1 < i_2 < \cdots < i_j$ of k.

(C) The number satisfying $1 \leqslant t \leqslant a_i \leqslant w$ for all i is the case $h_i = t$, $p_i = w$ for all i,

$$F(n, k; t, w; \cdots; t, w) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \binom{n-k(t-1)+j(t-w-1)-1}{k-1}$$

while

(D) the number satisfying $1 \leqslant t \leqslant a_i$ is (C) with $w = n$ or (A) wirh $h_i = t$,

$$F(n, k; t, n; \cdots; t, n) = \binom{n-k(t-1)-1}{k-1}.$$

(E) The number satisfying $1 \leqslant a_i \leqslant w$ is (C) with $t = 1$ or (B) with $p_i = w$,

$$F(n, k; 1, w; \cdots; 1, w) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \binom{n-jw-1}{k-1}.$$

In the case $w = 2$ it is easy to obtain another expression for this number, $\binom{k}{n-k}$, so

$$\sum_{j=0}^{k} (-1)^j \binom{k}{j} \binom{n-2j-1}{k-1} = \binom{k}{n-k} .$$

Corresponding restricted combinations. We now give the corresponding restricted combinations to the above examples using the correspondence described in Section 2. The number of r-combinations (2.1) of m satisfying for fixed $1 \leqslant h_i \leqslant p_i \leqslant m,\ i = 1, 2, \cdots, r+1$ the conditions

(3.3) $$h_1 \leqslant x_1 \leqslant p_1, \quad m - (p_{r+1} - 1) \leqslant x_r \leqslant m - (h_{r+1} - 1)$$

and

(3.4) $$h_{i+1} \leqslant x_{i+1} - x_i \leqslant p_{i+1}, \quad i = 1, 2, \cdots, r-1$$

is equal to $F(m+1, r+1; h_1, p_1; \cdots; h_{r+1}, p_{r+1})$. We consider now some special cases.

(F) The number of r-combinations satisfying conditions (3.4) only is obtained by putting $h_1 = h_{r+1} = 1$, $p_1 = p_{r+1} = m$.

(G) The number of combinations satisfying

$$h_1 \leqslant x_1, \quad 1 \leqslant x_r \leqslant m - (h_{r+1} - 1) \quad \text{and} \quad h_{i+1} \leqslant x_{i+1} - x_i, \quad i = 1, \cdots, r-1$$

is by using (A) equal to

$$\binom{m + r + 1 - h_1 - h_2 - \cdots - h_{r+1}}{r} .$$

(H) The number satisfying $h_{i+1} \leqslant x_{i+1} - x_i,\ i = 1, \cdots, r-1$ is (G) with $h_1 = h_{r+1} = 1$,

$$\binom{m + r - 1 - h_2 - h_3 - \cdots - h_r}{r} .$$

(I) The number satisfying

$$x_1 \leqslant p_1, \quad x_r \geqslant m - (p_{r+1} - 1) \quad \text{and} \quad x_{i+1} - x_i \leqslant p_{i+1}, \quad i = 1, \cdots, r-1$$

is equal to $F(m+1, r+1; 1, p_1; \cdots; 1, p_{r+1})$ while the number of combinations satisfying $x_{i+1} - x_i \leqslant p_{i+1}$ $i = 1, \cdots, r-1$, is given by the expression in (B) with $n - 1 = m,\ k - 1 = r$, and $p_1 = p_{r+1} = m$.

(J) The number satisfying

(3.5) $$t \leqslant x_1 \leqslant w, \quad m - (w-1) \leqslant x_r \leqslant m - (w-1)$$

and

$$t \leqslant x_{i+1} - x_i \leqslant w, \quad i = 1, \cdots, r-1$$

is given in (C) with $n - 1 = m,\ k - 1 = r$,

$$\sum_{j=0}^{r+1} (-1)^j \binom{r+1}{j} \binom{m - (r+1)(t-1) + j(t-w-1)}{r} .$$

(K) The number satisfying (3.5) only is equal to (3.2) with

$$n - 1 = m, \quad k - 1 = r, \quad h_1 = h_{r+1} = 1 \quad \text{and} \quad p_1 = p_{r+1} = m,$$

$$\sum_{j=0}^{r-1} (-1)^j \binom{r-1}{j} \binom{m - (r-1)(t-1) - j(1+w-t)}{r} .$$

(L) The number satisfying $t \leqslant x_{i+1} - x_i$, is (K) with $w = m$, or (H) with $h_2 = h_3 = \cdots = h_r = t$, is

$$\binom{m - (r-1)(t-1)}{r}$$

while in the case $t = 2$, no two consecutive elements in a combination, the above reduces to the familiar number

$$\binom{m - r + 1}{r} .$$

(M) The number satisfying $x_{i+1} - x_i \leqslant w$ is (K) with $t = 1$,

$$\sum_{j=0}^{r-1} (-1)^j \binom{r-1}{j} \binom{m-jw}{r} .$$

4. COMBINATIONS BY NUMBER AND LENGTH OF PARTS

Using correspondence (1.6) the number of $(n-k)$-combinations of $n-1$ with the length of each part less than or equal to $w-1$ is given by the expression in case (E) of Section 3. Putting $n=m+1,\ k=m-r+1$, the number of r-combinations of m with each part not greater than $w-1$ is equal to

$$\sum_{i=0}^{m-r+1} (-1)^i \binom{m-r+1}{i} \binom{m-iw}{m-r} . \tag{4.0}$$

More generally we consider the following' Given a set of q restrictions

$$A = (A_1, \cdots, A_q), \qquad A_j = \{ 2 \leqslant a_{j1} < a_{j2} < \cdots \} , \tag{4.1}$$

denote by $F_q(n, k; A)$ the number of k-compositions of n such that,

(4.2 a) $a_{i_j} \in A_j,\ j = 1, 2, \cdots, q$, for some q-combination $i_1 < i_2 < \cdots < i_q$ of $\{1, 2, \cdots k\}$.

(4.2 b) $a_i = 1$, for the remaining $k-q$ indices i.

Then

$$F_q(n, k; A) = \binom{k}{q} F(n-k+q, q; A) \tag{4.3}$$

or

$$F_q(n, k; A) = \binom{k}{q} F(n-k, q; B), \quad \text{where} \quad B = (B_1, \cdots, B_q), \tag{4.4}$$

$$B_j = \{ 1 \leqslant a_{j1} - 1 < a_{j2} - 1 < \cdots \}, \quad j = 1, \cdots, q .$$

Let a k-composition of n be given and suppose exactly q of the a_i,

$$a_{i_1}, a_{i_2}, \cdots, a_{i_q}, \quad i_1 < i_2 < \cdots < i_q ,$$

are each $\geqslant 2$. Using (1.6), to this k-composition of n corresponds a unique $(n-k)$-combination of $n-1$ with exactly q parts, the length of the j^{th} part (reading from left to right) being $a_{i_j} - 1,\ j = 1, 2, \cdots, q$. Hence $F_q(n, k; A)$ is the number of $(n-k)$-combinations of $(n-1)$ with exactly q parts, the length of the j^{th} equal to $a_j \in A_j$, $j = 1, \cdots, q$.

For convenience putting $k=m-r+1,\ n=m+1$, the number of r-combinations of m with the length of the j^{th} part equal to $a_j \in A_j$ is by substituting in (4.3) and (4.4), equal to

$$F_q(m+1, m-r+1; A) = \binom{m-r+1}{q} F(r+q, q; A) \tag{4.5}$$

or

$$F_q(m+1, m-r+1; A) = \binom{m-r+1}{q} F(r, q; B), \quad B \text{ given in (4.4).} \tag{4.6}$$

For fixed $1 \leqslant h_i \leqslant p_i \leqslant m$ and reading the parts from left to right it follows that the number of r-combinations of m having exactly q parts (or $r-q$ successions) and satisfying the restrictions,

$$h_i \leqslant \text{length of the } i^{th} \text{ part} \leqslant p_i, \quad i = 1, \cdots, q, \tag{4.7}$$

is equal to

$$\binom{m-r+1}{q} F(r, q; h_1, p_1; \cdots; h_q, p_q). \tag{4.8}$$

We consider now some special cases of (4.7). The number of combinations with exactly q parts such that the length of each part is greater or equal to t and less than or equal to w is the number (4.8) with $h_i = t,\ p_i = w$ for all i,

$$\binom{m-r+1}{q} \sum_{j=0}^{q} (-1)^j \binom{q}{j} \binom{r - q(t-1) + j(t-w-1) - 1}{q-1} , \tag{4.9}$$

while the number with each part $\geqslant t$ is equal to

$$(4.10)\qquad \binom{m-r+1}{q}\binom{r-q(t-1)-1}{q-1},$$

and the number with each part $\leqslant w$ is

$$(4.11)\qquad \binom{m-r+1}{q}\sum_{j=0}^{q}(-1)^j\binom{q}{j}\binom{r-jw-1}{q-1}.$$

Summing (4.11) over all $q \geqslant 1$ and using Vandermonde's Theorem, the number of combinations with each part $\leqslant w$ (and no restriction on the number of parts) is equal to

$$(4.12)\qquad \sum_{j=0}^{m-r+1}(-1)^j\binom{m-r+1}{j}\binom{m-j(w+1)}{m-r}$$

in agreement with (4.0) where each part is $\leqslant w-1$.

Thus we may enumerate a large class of restricted combinations using the above method. One further example is that each part is of even (odd) length while another is that the length is a multiple of a fixed number.

5. RECURRENCE RELATIONS

Denote k restrictions $A_1, \cdots, A_k$ by

$$(5.1)\qquad A^k = (A_1, \cdots, A_k), \qquad A_i = \left\{ 0 < a_{i1} < a_{i2} < \cdots \right\}, \quad i = 1, \cdots, k,$$

Then

$$(5.2)\quad F(n,k;A^k) = \sum_{\substack{a_1+\cdots+a_k=n \\ a_i\in A_i}} 1 = \sum_{\substack{a_k\in A_k \\ a_k\leqslant n}} \;\sum_{a_1+\cdots+a_{k-1}=n-a_k} = \sum_{\substack{a_k\in A_k \\ a_k\leqslant n}} F(n-a_k, k-1; A^{k-1}).$$

For the particular restrictions $1 \leqslant h_i \leqslant a_i \leqslant p_i$, i.e.,

$$(5.3)\qquad A_i = \left\{ h_i, h_i+1, \cdots, p_i \right\}, \qquad i = 1, \cdots, k,$$

we have

$$(5.4)\qquad F(n,k;A^k) = \sum_{h_k\leqslant a_k\leqslant p_k} F(n-a_k, k-1; A^{k-1})$$

$$= F(n-h_k, k-1; A^{k-1}) + \sum_{h_k\leqslant j\leqslant p_k-1} F(n-1-j, k-1; A^{k-1})$$

$$= F(n-h_k, k-1; A^{k-1}) + F(n-1, k; A^k) - F(n-1-p_k, k-1; A^{k-1}),$$

$$(F(n,k;A^k) = 0,\ n \leqslant 0)$$

with $F(n,k;A^k)$ the same as $F(n,k;p_1,h_1;\cdots;p_k,h_k)$ of (3.2). In the case $h_i = t$ and $p_i = n$, the number of compositions with each part of length not less than t, denoted by $F(n,k;\geqslant t)$ is

$$(5.5)\qquad F(n,k;\geqslant t) = \sum_{j=t}^{n-(k-1)t} F(n-j, k-1; \geqslant t) = F(n-t, k-1; \geqslant t) + F(n-1, k; \geqslant t).$$

Denoting by $F(n,k;\leqslant w)$ the number when $1 \leqslant a_i \leqslant w$, and using (5.4) with $h_i = 1$ and $p_i = w$ for all i,

$$(5.6)\quad F(n,k;\leqslant w) = \sum_{j=1}^{w} F(n-j, k-1; \leqslant w) = F(n-1, k-1; \leqslant w) + F(n-1, k; \leqslant w) - F(n-1-w, k-1; \leqslant w).$$

If we wish to consider compositions of n with given restrictions but with the number of parts not specified, then of course we simply sum over k. That is

$$G(n;A) = \sum_{k=1}^{n} F(n,k;A^k). \tag{5.7}$$

The generating function is

$$\sum_n G(n;A)x^n = \sum_k (x^{a11} + x^{a12} + \cdots)(x^{a21} + x^{a22} + \cdots)\cdots(x^{ak1} + x^{ak2} + \cdots).$$

For example, the number of compositions of n with each part not less than t, is by summing the expression in (D) of Section 3 over all k,

$$G(n;\geqslant t) = \sum_{k=1}^{\left[\frac{n}{t}\right]} \binom{n-k(t-1)-1}{k-1} \tag{5.8}$$

and satisfies the relation

$$G(n;\geqslant t) = G(n-t;\geqslant t) + G(n-1;\geqslant t). \tag{5.9}$$

In the case $t = 2$, $G(n;\geqslant 2)$ is the $(n-1)^{th}$ Fibonacci number, since $G(n;\geqslant 2) = 1$ or each of $n = 2,3$. The number with each part of length not greater than $w < n$ is by summing the expression of (E) in Section 3 over all k,

$$G(n;\leqslant w) = \sum_{k=1}^{n-w} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \binom{n-jw-1}{k-1} \tag{5.10}$$

and satisfies the relation

$$G(n;\leqslant w) = \sum_{i=1}^{w} G(n-i;\leqslant w) = 2G(n-1;\leqslant w) - G(n-1-w;\leqslant w). \tag{5.11}$$

In the case $w = 2$,

$$F(n;\leqslant 2) = \sum_{i=0}^{\left[\frac{n}{2}\right]} \binom{n-i}{i} \quad ,$$

and the above relation reduces to $G(n;\leqslant 2) = G(n-1;\leqslant 2) + G(n-2;\leqslant 2)$, $G(n;\leqslant 2)$ being the $(n+1)^{th}$ Fibonacci number since $G(n;\leqslant 2) = 1, 2$ for $n = 1, 2$, respectively.

We may obtain relations for the number counting restricted combinations by considering the number $F(n,k;A^k)$ which counts the corresponding restricted compositions.

6. CIRCULAR COMPOSITIONS AND COMBINATIONS

A (linear) composition (1.1) can be seen as a display of the integers $1, 2, \cdots, n$ in a line, with $k-1$ "dividers," no two dividers adjacent, which yield the k parts:

$$1, 2, \cdots, a_1/a_1+1, a_1+2, \cdots, a_1+a_2/\cdots/a_1+\cdots+a_{k-1}+1, \cdots, n. \tag{6.1}$$

The length of the i^{th} part (from left to right) is equal to a_i. For example the 4-composition of 9

$$2+3+1+3 = 9 \tag{6.2}$$

is seen as

$$12/345/6/789. \tag{6.3}$$

Analogously, a *circular* k-*composition of* n is a display of $1, 2, \cdots, n$ in a circle with k "dividers," no two dividers adjacent, yielding k parts each of length greater or equal to 1. We may illustrate a circular k-composition of n as

$$\tag{6.4} \underbrace{\overleftrightarrow{b, b+1, \cdots, n-1, n, 1, 2, \cdots, c}^{\,a_1}}_{\text{1st part}} / \underbrace{\overleftrightarrow{c+1, c+2, \cdots, c+a_2}^{\,a_2}}_{\text{2nd part}} / \cdots / \underbrace{\overleftrightarrow{b-a_k, \cdots, b-2, b-1}^{\,a_k}}_{k^{th}\text{ part}} /$$

placed on a circle in a clockwise direction with the integer 1 always belonging to the first part, i.e.,

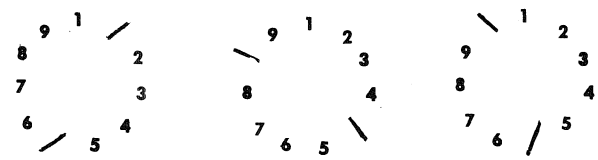
$$c \geq 1, \quad c+n-(b-1) = a_1, \quad a_i \geq 1.$$

Clearly the number of circular k-compositions (6.4) is equal to

$$\sum_{a_1+\cdots+a_k=n} a_1 = \binom{n}{k} .$$

For example,

(6.5)



or written as

(6.6) 67891/2345/, 91234/5678/, 12345/6789/,

recpectively, are three of the $\binom{9}{2}$ circular 2-compositions of 9.

To each circular composition (6.4) there corresponds a unique sequence placed on a circle in a clockwise direction,

$$\tag{6.7} 00\overset{*}{0} \cdots 01/000 \cdots 01/ \cdots /000 \cdots 01/$$

of $n-k$ 0's and k 1's with the 0 or 1 in the first part corresponding to the integer 1 of the composition marked by "*." Replacing the 1's by 0's and 0's by 1's in (6.7) we have a dual representation of the composition,

$$\tag{6.8} 11\overset{*}{1} \cdots 10/111 \cdots 10/ \cdots /111 \cdots 10/$$

of $n-k$ 1's and k 0's. We will call (6.7) and (6.8) "circular" sequences. For example, the circular sequences corresponding to each of (6.6), respectively, by use of (6.7) are

$$0000\overset{*}{1}/0001/, \qquad 0\overset{*}{0}001/0001/, \qquad \overset{*}{0}0001/0001/,$$

and by use of (6.8) are, respectively,

$$1111\overset{*}{0}/1110/, \qquad 1\overset{*}{1}110/1110/, \qquad \overset{*}{1}1110/1110/.$$

As earlier, consider the restrictions

$$A = (A_1, \cdots, A_k), \quad A_i = \{1 \leq a_{i1} < a_{i2} < \cdots\}, \quad i = 1, \cdots, k,$$

where each A_i is some given subset of $\{1, 2, 3, \cdots\}$. Denote by $C(n, k; A)$ the number of circular compositions (6.4) with $a_i \in A_i$, $i = 1, \cdots, k$. That is

$$C(n, k; A) = \sum_{\substack{a_1+\cdots+a_k=n \\ a_i \in A_i}} a_i .$$

Then the generating function is,

$$\sum_n C(n, k; A)x^n = (a_{11}x^{a_{11}} + a_{12}x^{a_{12}} + \cdots)(x^{a_{21}} + x^{a_{22}} + \cdots)(x^{a_{31}} + x^{a_{32}} + \cdots) \cdots (x^{a_{k1}} + x^{a_{k2}} + \cdots).$$

Checking for the case $A_i = \{1, 2, 3, \cdots\}$ for all i,

$$\sum_{n=k} C(n,k;A)x^n = (x+2x^2+3x^3+\cdots)(x+x^2+x^3+\cdots)^{k-1} = x(1-x)^{-2}x^{k-1}(1-x)^{-(k-1)}$$

$$= \sum_{n=k} \binom{n}{k} x^n .$$

An example of the use of the above generating function is obtained by taking $A_i = \left\{ h_i, h_i+1, h_i+2, \cdots \right\}$, $i = 1, \cdots, k$ and letting $h = h_1 + \cdots + h_k$,

$$\sum_{n=h} C(n,k;A) = (h_1x^{h_1} + (h_1+1)x^{h_1+1} + \cdots) \prod_{i=2}^{k} (x^{h_i} + x^{h_i+1} + \cdots)$$

$$= (h_1 - h_1x + x)x^{h_1}(1-x)^{-2}x^{h-h_1}(1-x)^{-(k-1)}$$

$$= (h_1 - h_1x + x)x^h \sum_{i=0} \binom{k+i}{k} x^i$$

$$= h_1x^h + \sum_{i=0} \left[h_1 \binom{k+i+1}{k} + (1-h_1)\binom{k+i}{k} \right] x^{h+i+1}$$

$$= h_1x^h + \sum_{i=0} \frac{h_1k+i+1}{k+i+1} \binom{k+i+1}{k} x^{h+i+1}$$

$$= \sum_{n=h} \frac{h_1k+n-h}{k+n-h} \binom{k+n-h}{k} x^n ,$$

and hence the number of compositions (6.4) with $1 \leqslant h_i \leqslant a_i$, $i = 1, \cdots, k$ is

$$\frac{h_1k+n-h}{k+n-h}\binom{k+n-h}{k}, \qquad h = h_1 + \cdots + h_k . \tag{6.9}$$

We now consider a more general example which includes as a special case (6.9). Given $1 \leqslant h_i \leqslant p_i \leqslant m$, the number of circular compositions (6.4) satisfying $h_i \leqslant a_i \leqslant p_i$, $i = 1, 2, \cdots, k$ is

$$C(n,k;h_1,p_1;\cdots;h_k,p_k) = \sum_{\substack{a_1+\cdots+a_k=n \\ h_i \leqslant a_i \leqslant p_i}} a_1 = \sum_{a_1=h_1}^{p_1} a_1 \sum_{\substack{a_2+\cdots+a_k=n-a_1 \\ h_i \leqslant a_i \leqslant p_i}} 1 \tag{6.10}$$

$$= \sum_{a_1=h_1}^{p_1} a_1 F(n-a_1, k-1; h_2, p_2; \cdots; h_k, p_k),$$

where $F(n,k;h_2,p_2;\cdots;h_k,p_k)$ is given by (3.2). Using the identity

$$\sum_{i=m}^{n} i\binom{x+k-2-i}{k-2} = \binom{x+k-m}{k}\frac{x+km-m}{x+k-m} - \binom{x+k-n-1}{k}\frac{x+(n+1)(k-1)}{x+k-n-1} \tag{6.11}$$

$$= \binom{x+k-m}{k} + \binom{x+k-m-1}{k-1}(m-1) - \binom{x+k-n-1}{k} - \binom{x+k-n-2}{k-1}n$$

and (3.2), (6.10) reduces to

$$C(n, k; h_1, p_1; \cdots, h_k p_k)$$

$$= \sum_{j=0}^{k-1} (-1)^j \Sigma^* \left[\binom{x+k-h_1}{k} \frac{x+kh_1-h_1}{x+k-h_1} - \binom{x+k-p_1-1}{k} \frac{x+(k-1)(1+p_1)}{x+k-p_1-1} \right]$$

(6.12)

$$= \sum_{j=0}^{k-1} (-1)^j \Sigma^* \left[\binom{x+k-h_1}{k} + \binom{x+k-h_1-1}{k-1} (h_1-1) - \binom{x+k-p_1-1}{k} - \binom{x+k-p_1-2}{k-1} p_1 \right],$$

where

$$h = h_1 + \cdots + h_k, \quad x = n - h + h_1 - j - (p_{i_1} - h_{i_1}) - \cdots - (p_{i_j} - h_{i_j}) \quad \text{for } j > 0,$$

$x = n - h + h_1$ for $j = 0$ and the summation Σ^* is taken over all j combinations

$$i_1 < i_2 < \cdots < i_j \quad \text{of } \{2, 3, \cdots, k\}.$$

We consider now some of the many special cases of (6.12). The number of circular compositions satisfying:

(A) $h_i \leqslant a_i$, $i = 1, 2, \cdots, k$ is (6.12) with $p_i = n$ for all i, is the first term of second last expression for $j = 0$,

$$\frac{n-h+kh_1}{n-h+k} \binom{n-h+k}{k},$$

in agreement with (6.9),

(B) $a_i \leqslant p_i$, $i = 1, \cdots, k$ is (6.12) with $h_i = 1$ for all i,

$$\sum_{j=0}^{k-1} (-1)^j \Sigma^* \left[\binom{y}{k} - \binom{y-p_1}{k} - \binom{y-p_1-1}{k-1} p_1 \right]$$

$$= \sum_{j=0}^{k-1} (-1)^j \Sigma^* \left[\binom{y}{k} - \binom{y-p_1}{k} \frac{y+p_1(k-1)}{y-p_1} \right],$$

where $y = n - (p_{i_1} + \cdots + p_{i_j})$, the summation Σ^* taken over all j-combinations $i_1 < \cdots < i_j$ of $\{2, \cdots, k\}$ for $j \geqslant 1$ and $y = n$ when $j = 0$.

(C) $h_1 \leqslant a_1 \leqslant p_1$ and $t \leqslant a_i \leqslant w$ for $i = 2, 3, \cdots, k$ is (6.12) with $h_i = t$, $p_i = w$, $i = 2, \cdots, k$,

$$\sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} \left[\binom{n-(k-1)t+k-h_1-j(1+w-t)}{k} \frac{n-(k-1)(t-h_1)-j(1+w-t)}{n-(k-1)t+k-h_1-j(1+w-t)} \right.$$

$$\left. - \binom{n-(k-1)(t-1)-p_1-j(1+w-t)}{k} \frac{n-(k-1)(t-1-p_1)-j(1+w-t)}{n-(k-1)(t-1)-p_1-j(1+w-t)} \right]$$

$$= \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} \left[\binom{n-(k-1)(t-1)-j(1+w-t)-h_1+1}{k} \right.$$

$$+ \binom{n-(k-1)(t-1)-j(1+w-t)-h_1}{k-1} (h_1-1)$$

$$\left. - \binom{n-(k-1)(t-1)-j(1+w-t)-p_1}{k} + \binom{n-(k-1)(t-1)-j(1+w-t)-p_1-1}{k-1} p_1 \right].$$

(D) $t \leqslant a_i \leqslant w$ is case (C) with $h_1 = t$, $p_1 = w$,

(E) $t \leqslant a_i$ for all i, is case (D) with $w = n$ or case (A) with $h_i = t$ for all i,

$$\frac{n}{n-k(t-1)} \binom{n-k(t-1)}{k}.$$

(F) $a_i \leqslant w$ for all i is case (D) with $t = 1$ or case (B) with $p_i = w$.

$$\sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} \left[\binom{n-jw}{k} - \binom{n-w(j+1)}{k} \frac{n-w(j+1)+wk}{n-w(j+1)} \right]$$

$$= \binom{n}{k} + \sum_{i=1}^{k} (-1)^i \left[\binom{k-1}{i} \binom{n-iw}{k} + \binom{k-1}{i-1} \binom{n-iw}{k} \frac{n-iw+wk}{n-iw} \right]$$

$$= \frac{n}{k} \sum_{i=0}^{k} (-1)^i \binom{k}{i} \binom{n-iw-1}{k-1},$$

and in the case $w = 2$ another expression is

$$\frac{n}{k} \binom{k}{n-k},$$

see case (E) of Section 3.

To obtain recurrence relations we proceed as follows. Let $A^k = (A_1, \cdots, A_k)$. Then for $k \geq 2$,

$$(6.13) \quad C(n,k;A^k) = \sum_{\substack{a_1+\cdots+a_k=n \\ a_i \in A_i}} a_1 = \sum_{\substack{a_k \in A_k \\ a_k \leq n}} \sum_{a_1+\cdots+a_{k-1}=n-a_k} a_1 = \sum_{\substack{a_k \in A_k \\ a_k \leq n}} C(n-a_k, k-1; A^{k-1}).$$

This is the same as that for the linear case (5.2) with different initial values. For the particular restrictions $1 \leq h_i \leq a_i \leq p_i$, i.e.,

$$A_i = \{h_i, h_i+1, \cdots, p_i\}, \qquad i = 1, \cdots, k,$$

we have

$$(6.14) \quad C(n,k;A^k) = \sum_{h_k \leq a_k \leq p_k} C(n-a_k, k-1; A^{k-1})$$

$$= C(n-h_k, k-1; A^{k-1}) + \sum_{h_k \leq j \leq p_k - 1} C(n-1-j, k-1; A^{k-1})$$

$$= C(n-h_k, k-1; A^{k-1}) + C(n-1, k; A^k) - C(n-1-p_k, k-1; A^{k-1}),$$

$$(C(n,k;A^k) = 0, \quad n \leq 0).$$

The number of circular compositions with each $a_i \geq t$, denoted by $C(n,k;\geq t)$ and given by the expression in case (E) above satisfies the relation

$$(6.15) \qquad C(n,k;\geq t) = C(n-t, k-1; \geq t) + C(n-1, k; \geq t).$$

Denoting by $C(n,k;\geq w)$ the number when $1 \leq a_i \leq w$ then the expression is given in case (F) above and satisfies the relation

$$(6.16) \qquad C(n,k;\leq w) = \sum_{j=1}^{w} C(n-j, k-1; \leq w)$$

$$= C(n-1, k-1; \leq w) + C(n-1, k; \leq w) - C(n-1-w, k-1; \leq w).$$

Summing (6.15) over all k the number of circular compositions with each part not less than t is

$$(6.17) \qquad D(n;\geq t) = \sum_{k=0}^{\left[\frac{h}{t}\right]} \frac{n}{n-k(t-1)} \binom{n-k(t-1)}{k}$$

and

(6.18) $$D(n; \geqslant t) = D(n-t; \geqslant t) + D(n-1; \geqslant t).$$

In the case $t = 2$, the above relation reduces to

$$D(n; \geqslant 2) = D(n-2; \geqslant 2) + D(n-1; \geqslant 2)$$

and $D(n; \geqslant 2)$ is the Lucas number having values 1, 3 for $n = 1, 2$, respectively. Summing (6.16) over all k the number $D(n; \leqslant w)$ of circular compositions with each part not greater than w is

(6.19) $$D(n; \leqslant w) = \sum_{k=1} \frac{n}{k} \sum_{i=0}^{k} (-1)^i \binom{k}{i} \binom{n-iw-1}{k-1}$$

and satisfies the relation

(6.20) $$D(n; \leqslant w) = \sum_{j=1}^{w} D(n-j; \leqslant w).$$

In the case $w = 2$, $D(n; \geqslant 2)$ is also the Lucas number with $D(n; \geqslant 2)$ having values 1, 3 for $n = 1, 2$, respectively.

Given a set of q restrictions

$$A = (A_1, \cdots, A_q), \qquad A_j = \left\{ 2 \leqslant a_{j1} \leqslant a_{j2} \leqslant \cdots \right\},$$

denote by $C_q(n, k; A)$ the number of circular compositions (6.4) such that

(a) $a_{i_j} \in A_j$, $j = 1, 2, \cdots, q$ for some q-combination

$$i_1 < i_2 < \cdots < i_q \text{ of } \left\{1, 2, \cdots, k\right\},$$

(b) $a_1 = 1$ for the remaining $k - q$ indices i.

Then by partitioning the compositions into those with $a_1 = 1$ and $a_1 > 1$

(6.21) $$\begin{aligned} C_q(n, k; A) &= \binom{k-1}{q} F(n-k+q, q; A) + \binom{k-1}{q-1} C(n-k+q, q; A) \\ &= \binom{k-1}{q} F(n-k, q; B) + \binom{k-1}{q-1} C(n-k, q; B) + \binom{k-1}{q-1} F(n-k, q; B) \\ &= \binom{k}{q} F(n-k, q; B) + \binom{k-1}{q-1} C(n-k, q; B), \end{aligned}$$

where

$$B = (B_1, \cdots, B_q), \qquad B_j = \left\{1 \leqslant a_{j1} - 1 \leqslant a_{j2} - 1 \leqslant \cdots\right\}, \; j = 1, \cdots, q$$

and $F(n, k; A)$ is the number of restricted (linear) compositions discussed earlier.

7. CIRCULAR COMBINATIONS

A *circular k-combination of* n is a set of k integers

(7.1) $$x_1 < x_2 < \cdots < x_k$$

chosen from the integers $1, 2, \cdots, n$ displayed in a circle. That is we consider 1 and n to be consecutive. For example the circular 6-combination 1, 3, 4, 5, 8, 9 of 9 has parts (891) and (345) each of length 3 while the same (linear) 6-combination has parts (1), (345), (89). Of course, the number $\binom{n}{k}$ of (linear) k-combinations of n is equal to the number of circular k-combinations of n. A succession here is a pair x_i, x_{i+1} with $x_{i+1} - x_i = 1$ with $n, 1$ also considered a succession. As before if a combination (7.1) has q parts it has $k - q$ successions. As before to each circular combination (7.1) corresponds a unique sequence of k 1's and $n - k$ 0's.

(7.2) $$e_1^*, e_2, \cdots, e_n$$

with

$$e_i = \begin{cases} 1 & \text{if } i \text{ is in the combination,} \\ 0 & \text{if } i \text{ is not in the combination.} \end{cases}$$

We shall think of the sequence (7.2) placed on a circle in a clockwise direction. Hence the "circular" sequence (7.2) corresponds to the circular sequence (6.7) by agreeing to let e_1 correspond to the element of (6.7) marked by a *. To a circular composition (6.4) corresponds a unique circular combination (7.2) with

$$n-(x_k-x_1) = a_1$$

$$x_{i+1}-x_i = a_i \quad \text{for} \quad i = 1,2,\cdots,k-1.$$

Thus the number of combinations (7.1) satisfying the restrictions

$$n-(x_k-x_1) \in A_1 \quad \text{and} \quad x_{i+1}-x_i \in A_i \quad \text{for} \quad i = 1,2,\cdots,k-1,$$

where the A_i are given by (6.7), is simply the number $C(n, k; A)$ of Section 6. For example the number of combinations satisfying

$$h_1 \leqslant n-(x_k-x_1) \leqslant p_1 \quad \text{and} \quad t \leqslant x_{i+1}-x_i \leqslant w \quad \text{for} \quad i = 1,2,\cdots,k-1$$

is the expression of case (C) of Section 6 and is in agreement with Moser and Abramson [1969 a, expression (14) for $C_{n,k}(t, w; h_1, p_1)$].

Using the dual representation (6.8) and (7.2) we have a one-one correspondence between the circular compositions (6.4) and circular $(n-k)$-combinations of n. For example the number of circular $(n-k)$-combinations of n with each part of length not greater than $w-1$ is the number of circular compositions with $a_i \leqslant w$ given in case (F) of Section 6. Putting $n = m$ and $k = m-r$ the number of circular r-combinations of m is

$$\text{(7.3)} \qquad \frac{m}{m-r} \sum_{i=0}^{m-r} (-1)^i \binom{m-r}{i} \binom{m-iw-1}{m-r-1}$$

in agreement with Moser and Abramson [1969 a, expression (29)].

More generally the number of circular r-combinations of m having exactly q parts, or $r-q$ successions, the length of the j^{th} part (reading in a clockwise direction with the first part that part containing the smallest integer greater than or equal to 1) equal to $a_j - 1$, $a_j \in A_j$, $j = 1, 2, \cdots, q$ is $C_q(m, m-r; A)$ given by (6.21).

For example letting $A_i = \{t+1, t+2, \cdots\}$ for all i the number of circular r-combinations of m with exactly q parts and with each part of length not less than t is by using (6.21), (D) of Section 3 and (E) of Section 6,

$$\text{(7.4)} \qquad \begin{aligned} C_q(m, m-r; A) &= \binom{m-r}{q} F(r,q;B) + \binom{m-r-1}{q-1} C(r,q;B) \\ &= \binom{m-r}{q}\binom{r-q(t-1)-1}{q-1} + \binom{m-r-1}{q-1}\binom{r-q(t-1)}{q} \frac{r}{r-q(t-1)} \\ &= \binom{m-r}{q}\binom{r-q(t-1)-1}{q-1} \frac{m}{m-r} \quad . \end{aligned}$$

The number with exactly q parts each of length not greater than w is obtained by taking $B_i = \{1, 2, \cdots, w\}$ for all i and using (E) of Section 3 and (F) of Section 6,

$$\text{(7.5)} \qquad \begin{aligned} C_q(m, m-r; A) &= \binom{m-r}{q} F(r,q;B) + \binom{m-r-1}{q-1} C(r,q;B) \\ &= \frac{m}{m-r}\binom{m-r}{q} \sum_{i=0} (-1)^i \binom{q}{i}\binom{r-iw-1}{q-1} \\ &= \frac{m}{m-r} \sum_{i=0} (-1)^i \binom{m-r}{i}\binom{m-r-i}{q-i}\binom{r-iw-1}{q-1} . \end{aligned}$$

Summing (7.5) over all q we obtain the number of circular combinations of m with each part of length not greater than w.

$$\frac{m}{m-r} \sum_{i=0}^{m-r} (-1)^i \binom{m-r}{i}\binom{m-i(w+1)-1}{m-r-1}$$

in agreement with (7.3) where a part is of length not greater than $w - 1$.

REFERENCES

Morton Abramson, 1971, "Combinations, Compositions and Occupancy Problems," *The Fibonacci Quarterly,* Vol. 9, No. 2 (April 1971), pp. 225–236.

Morton Abramson and W. O. J. Moser, 1976, "Linear and Ring Arrangements," *Portugaliae Math.* (to appear).

C. A. Church, Jr., 1966, "On a Lemma of M. Abramson," *Canadian Math. Bull.* (9), pp. 347–348.

C. A. Church, Jr., 1968, "Combinations and Successions," *Math. Mag.* (41), pp. 123–128.

C. A. Church, Jr., 1970, "On the Enumeration of Certain Triangular Arrays," *The Fibonacci Quarterly,* Vol. 8, No. 4 (Dec. 1970), pp. 235–241.

C. A. Church, Jr., 1971, "Combinations and Their Duals," *The Fibonacci Quarterly,* Vol. 9, No. 4 (Dec. 1971), pp. 505–509

C.A. Church, Jr., and H. W. Gould, 1967, "Problem of Terquem and an Extension of Fibonacci Numbers," *The Fibonacci Quarterly,* Vol. 5, No. 1 (Feb. 1967), pp. 59–68.

I. Kaplansky, 1943, Solution to the "Problème des Ménages," *Bull. Amer. Math. Soc.,* (49), pp. 784–785.

W. O. J. Moser and Morton Abramson, 1969 a, "Enumeration of Combinations with Restricted Differences and Cospan," *J. Comb. Theory,* (7), pp. 162–170.

W. O. J. Moser and Morton Abramson, 1969 b, "Generalizations of Terquem's Problem," *J. of Comb. Theory* (7), pp. 171–180.

J. Riordan, 1968, *An Introduction to Combinatorial Analysis,* Wiley, New York.

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ODE TO PASCAL'S TRIANGLE

Pascal. . . Pascal, you genius, you,
Constructed a triangle of powers of two.
Coefficients, and powers of eleven, by base ten,
A more useful aid, there's never been.
Head, tail, tail, head,
Answers from your rows are read.
Combinations and expectations, to my delight,
Can also be proved wrong or right.
With a little less effort and a little more ease,
I might have gotten thru this course in a breeze.
So, Pascal . . . Pascal, you rascal you.
Why did you limit it to powers of two?

. . . Bob Jones
Southern Baptist College
Blytheville, AR 72315

[See p. 455 for "Response."]