ADVANCED PROBLEMS AND SOLUTIONS

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Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-267 Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, California.

Show that

$$S(x) = \sum_{n=0}^{\infty} \frac{1}{kn+1} \frac{(knx)^n}{n!}$$

satisfies $S(x) = e^{xS^k(x)}$.

H-268 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Put

$$S_n(x) = \sum_{k=0}^{n} S(n,k)x^k,$$

where $S(n,k)$ denotes the Stirling number of the second kind defined by

$$x^n = \sum_{k=0}^{n} S(n,k)x(x-1)\cdots(x-k+1).$$

Show that

$$\left\{ \begin{aligned} xS_n(x) &= \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} S_{j+1}(x) \\ S_{n+1}(x) &= x \sum_{j=0}^{n} \binom{n}{j} S_j(x). \end{aligned} \right.$$

More generally evaluate the coefficients $c(n,k,j)$ in the expansion

$$x^k S_n(x) = \sum_{j=0}^{n+k} c(n,k,j) S_j(x) \qquad (k, n \geq 0).$$

SOLUTIONS

SYSTEMATIC WORK

H-244 Proposed by L. Carlitz and T. Vaughan, Durham, North Carolina and Greensboro, North Carolina.

Solve the system of equations

$$(*)\qquad x_j = a_j + \mu a_j \sum_{t=1}^{j-1} x_t + a_j x_j + \lambda a_j \sum_{t=j+1}^{n} x_t \quad (j = 1, 2, \cdots, n)$$

for $x = x_1 + x_2 + \cdots + x_n$, where $a_k \neq 0$ $(k = 1, 2, \cdots, n)$ and $\lambda \neq \mu$.

Solution by the Proposer.

For $j = 1$, (*) reduces to

$$x_1 = a_1 + a_1 x_1 + \lambda a_1 (x - x_1),$$

so that

$$\big(1 - (1-\lambda)a_1\big)x_1 = a_1(\lambda x + 1).$$

For $j = 2$, (*) becomes

$$x_2 = a_2 + \mu a_2 x_1 + a_2 x_2 + \lambda a_2 (x - x_1 - x_2),$$

so that

$$\big(1 - (1-\lambda)a_2\big)x_2 = a_2(\lambda x + 1) - a_2(\lambda - \mu)x_1 .$$

Hence

$$\big(1 - (1-\lambda)a_1\big)\big(1 - (1-\lambda)a_2\big)x_2 = a_2\big(1 - (1-\mu)a_1\big)(\lambda x + 1).$$

Similarly, for $j = 3$,

$$x_3 = a_3 + \mu a_3(x_1 + x_2) + a_3 x_3 + \lambda a_3(x - x_1 - x_2 - x_3).$$

After a little manipulation we get

$$\big(1 - (1-\lambda)a_1\big)\big(1 - (1-\lambda)a_2\big)\big(1 - (1-\lambda)a_3\big)x_3 = a_3\big(1 - (1-\mu)a_1\big)\big(1 - (1-\mu)a_2\big)(\lambda x + 1).$$

The general formula

$$(1)\qquad f_1(\lambda)f_2(\lambda)\cdots f_k(\lambda)x_k = a_k f_1(\mu)\cdots f_{k-1}(\mu)(\lambda x + 1) \qquad (1 \leqslant k \leqslant n),$$

where

$$f_k(\lambda) = 1 - (1-\lambda)a_k ,$$

is now easily proved by induction on k.

Returning to (*), we take $j = n$. Thus

$$x_n = a_n + \mu a_n(x - x_n) + a_n x_n ,$$

so that

$$\big(1 - (1-\mu)a_n\big)x_n(\mu x + 1).$$

For $j = n - 1$ we get

$$x_{n-1} = a_{n-1} + \mu a_{n-1}(x - x_n - x_{n-1}) + a_{n-1}x_{n-1} + \lambda a_{n-1}x_n .$$

This gives

$$\big(1 - (1-\mu)a_n\big)\big(1 - (1-\mu)a_{n-1}\big)x_{n-1} = a_{n-1}\big(1 - (1-\lambda)a_n\big)(\mu x + 1).$$

Similarly, for $j = n - 2$,

$$\big(1 - (1-\mu)a_n\big)\big(1 - (1-\mu)a_{n-1}\big)\big(1 - (1-\mu)a_{n-2}\big)x_{n-2} = a_{n-2}\big(1 - (1-\lambda)a_n\big)\big(1 - (1-\lambda)a_{n-1}\big)(\mu x + 1).$$

The general formula

$$(2)\qquad f_n(\mu)f_{n-1}(\mu)\cdots f_{n-k+1}(\mu)x_{n-k+1} = a_{n-k+1}f_n(\lambda)\cdots f_{n-k+2}(\lambda)(\mu x + 1) \quad (1 \leqslant k \leqslant n)$$

is easily proved by induction.

In (2) replace k by $n - k + 1$:

$$(3)\qquad f_n(\mu)f_{n-1}(\mu)\cdots f_k(\mu)x_k = a_k f_n(\lambda)\cdots f_{k+1}(\lambda)(\mu x + 1) \qquad (1 \leqslant k \leqslant n).$$

Comparing (3) with (1) we get

$$a_k \frac{f_1(\mu)\cdots f_{k-1}(\mu)}{f_1(\lambda)f_2(\lambda)\cdots f_k(\lambda)}(\lambda x + 1) = a_k \frac{f_n(\lambda)\cdots f_{k+1}(\lambda)}{f_n(\mu)f_{n-1}(\mu)\cdots f_k(\mu)}(\mu x + 1).$$

Since $a_k \neq 0$, it follows that

$$F_n(\mu)(\lambda x + 1) = F_n(\lambda)(\mu x + 1),$$

where

(5)
$$F_n(\lambda) = \prod_{k=1}^{n} f_k(\lambda) = \prod_{k=1}^{n} \big(1 - (1-\lambda)a_k\big).$$

Solving (4) for x, we get

(6)
$$x = \frac{F_n(\lambda) - F_n(\mu)}{\lambda F_n(\mu) - \mu F_n(\lambda)} .$$

Since, by (6),

$$x + 1 = \frac{(\lambda - \mu)F_n(\lambda)}{\lambda F_n(\mu) - \mu F_n(\lambda)} .$$

Hence (1) gives

(7)
$$x_k = a_k \frac{F_{k-1}(\mu)}{F_k(\lambda)} \frac{(\lambda - \mu)F_n(\lambda)}{\lambda F_n(\mu) - \mu F_n(\lambda)} \qquad (1 \leqslant k \leqslant n),$$

where

$$F_k(\lambda) = \prod_{j=1}^{k} f_k(\lambda) \qquad (1 \leqslant k \leqslant n).$$

PRODUCTIVE IDENTITY

H-245 Proposed by P. Bruckman, University of Illinois at Chicago, Chicago, Illinois.

Prove the identity

(1)
$$\sum_{k=0}^{n} \frac{x^{\frac{1}{2}k(k-1)}}{(x)_k(x)_{n-k}} = \frac{2\prod_{r=1}^{n-1}(1+x^r)}{(x)_n}, \qquad n = 1, 2, \cdots,$$

where

$$(x)_n = (1-x)(1-x^2)(1-x^3)\cdots(1-x^n), \qquad n = 1, 2, \cdots; \ (x)_0 = 1.$$

Solution by the Proposer.

Lemma 1. If

$$A(w,x) = \prod_{r=1}^{\infty} (1 + x^r w), \qquad \text{then} \qquad A(w,x) = \sum_{n=0}^{\infty} \frac{x^{\frac{1}{2}n(n+1)}}{(x)_n} w^n .$$

Proof. In a previously submitted proposed problem for this section [H-236], the author established the following identity:

(2)
$$f(z,y) = \prod_{r=1}^{\infty} (1 + y^{2r-1}z) = \sum_{n=0}^{\infty} \frac{y^{n^2}}{(y^2)_n} z^n .$$

Letting $y = x^{\frac{1}{2}}$, $z = wx^{\frac{1}{2}}$ in this identity, we find that the lemma is established, with $A(w,x) = f(wx^{\frac{1}{2}}, x^{\frac{1}{2}})$.

Lemma 2. If

$$B(w,x) = \prod_{r=1}^{\infty} (1 - x^r w)^{-1}, \qquad \text{then} \qquad B(w,x) = \sum_{n=0}^{\infty} \frac{x^n}{(x)_n} w^n .$$

Proof. This is equivalent to identity (7) in the above-mentioned problem. Now, consider the product

$$F(w,x) = A(w,x)B(w,x),$$

which is also equal to

$$\prod_{r=1}^{\infty} \frac{1 + x^r w}{1 - x^r w} .$$

We observe that

$$F(wx,x) = \prod_{r=1}^{\infty} \frac{1+x^{r+1}w}{1-x^{r+1}w} = \prod_{r=2}^{\infty} \frac{1+x^r w}{1-x^r w} = \frac{1-xw}{1+xw} F(w,x).$$

Now suppose

$$F(w,x) = \sum_{n=0}^{\infty} \theta_n(x) w^n .$$

We then have

$$(1-xw)\sum_{n=0}^{\infty} \theta_n(x) w^n = (1+xw)\sum_{n=0}^{\infty} \theta_n(x)(xw)^n ,$$

which yields the recursion

$$\theta_n(x) = \frac{x(1+x^{n-1})}{1-x^n}\,\theta_{n-1}(x), \qquad n = 1, 2, \cdots .$$

Since $F(0,x) = 1 = \theta_0(x)$, we readily obtain, by induction, that

$$\theta_n(x) = \frac{2x^n(1+x)(1+x^2)\cdots(1+x^{n-1})}{(x)_n}, \quad n = 1, 2, \cdots, \text{ with } \theta_0(x) = 1.$$

Hence,

$$F(w,x) = \prod_{r=1}^{\infty} \frac{1+x^r w}{1-x^r w} = 1 + 2\sum_{n=1}^{\infty} \frac{x^n(1+x)\cdots(1+x^{n-1})}{(x)_n}\, w^n . \tag{3}$$

However, since

$$F(w,x) = A(w,x)B(w,x) = \sum_{n=0}^{\infty} \frac{x^{\frac{1}{2}n(n+1)}}{(x)_n}\, w^n \cdot \sum_{n=0}^{\infty} \frac{x^n}{(x)_n}\, w^n ,$$

we also obtain the formula

$$F(w,x) = \sum_{n=0}^{\infty} w^n \sum_{k=0}^{n} \frac{x^{\frac{1}{2}k(k+1)}}{(x)_k}\,\frac{x^{n-k}}{(x)_{n-k}} . \tag{4}$$

Comparing coefficients of w in (3) and (4), we obtain for $n = 1, 2, \cdots$,

$$\frac{2x^n(1+x)\cdots(1+x^{n-1})}{(x)_n} = \sum_{k=0}^{n} \frac{x^{n+\frac{1}{2}k(k-1)}}{(x)_k(x)_{n-k}} .$$

Upon dividing each side by x^n, we find that (1) is established.

Also solved by P. Tracy and A. Shannon.

FIB, LUC, ET AL

H-246 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Put

$$F(m,n) = \sum_{i=0}^{m}\sum_{j=0}^{n} F_{i+j}F_{m-i+j}F_{i+n-j}F_{m-i+n-j}$$

$$L(m,n) = \sum_{i=0}^{m}\sum_{j=0}^{n} L_{i+j}L_{m-i+j}L_{i+n-j}L_{m-i+n-j} .$$

[Continued on page 473.]