ON POLYNOMIALS GENERATED BY TRIANGULAR ARRAYS

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In this paper we study a class of functions which we call Pascal functions, generated by the diagonals of triangular arrays, and discuss some of their properties. The Fibonacci polynomials become particular cases of Pascal functions, and so our results are of a fairly general nature.

1. DEFINITIONS AND GENERAL PROPERTIES

Consider a polynomial function in two variables, $p(x,y)$. It is defined to be a Pascal function of $(k-1)^{st}$ order if

$$p(x,y) = \sum_{m=0}^{[n/k]} a_m x^{n-km} y^m, \tag{1}$$

where the a_m are non-zero constants, and $[x]$ represents, for real x, the largest integer not exceeding x. Let us denote the set of all Pascal functions (polynomials) of k^{th} order by Π_k. (Note: k is a positive integer.)

One generalization of the famous Fibonacci polynomials is

$$F_0(x,y) = 0, \quad F_1(x,y) = 1, \quad F_{n+2}(x,y) = xF_{n+1}(x,y) + yF_n(x,y), \quad n = 0, 1, 2, \cdots.$$

We find that

$$F_n(x,y) \in \Pi_1, \qquad n = 0, 1, 2, 3, \cdots.$$

See Hoggatt and Long [1]. It is interesting to note that the following properties hold:

Lemma 1. If $p(x,y)$ and $p^*(x,y)$ are in Π_k, then $q(x,y)$ is in Π_k, where

$$q(x,y) = p(x,y)p^*(x,y).$$

This is the same as saying that Π_k is closed under multiplication.

If $p(x,y) \in \Pi_{k-1}$, and has an expansion as given in (1), then let $D(p) = n$. We then have

Lemma 2. If $p(x,y)$ and $p^*(x,y)$ are in Π_k, then

$$q(x,y) = p(x,y) + p^*(x,y)$$

is in Π_k if and only if $D(p) = D(p^*)$.

Lemma 3. If $p(x,y)$ is in Π_k, then

$$\frac{\partial p(x,y)}{\partial x} \quad \text{and} \quad \frac{\partial p(x,y)}{\partial y}$$

are in Π_k.

The three lemmas given above can be proved easily.

We define a sequence of functions

$$\left\{ p_n(x,y) \right\}_0^\infty \in \Pi_k$$

to be proper if

$$D(p_{n+1}) = D(p_n) + 1 \qquad \text{with} \qquad D(p_0) = D(p_1) = 0. \tag{2}$$

By a Pascal array we mean a triangular array of numbers represented in Fig. 1 below:

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$$\begin{array}{ccccc} c_{0,0} & & & & \\ c_{1,0} & c_{1,1} & & & \\ c_{2,0} & c_{2,1} & c_{2,2} & & \\ c_{3,0} & c_{3,1} & c_{3,2} & c_{3,3} & \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{array}$$

Figure 1

If now we replace every $c_{i,j}$ by $c_{i,j}x^iy^j$, and take the rising diagonal sums, where the rising diagonals have a slope k, we get a proper sequence in Π_k. Conversely, to every proper sequence in Π_k, we can associate a triangular array as in Fig. 1. Note that we can get infinitely many proper sequences from Fig. 1 as k varies, and all of these sequences for different values of k, we call "associated sequences." The triangular array which generates these sequences, is called their "associated array."

We now discuss some special properties of $p(x,y) \in \Pi_k$.

2. SOME SPECIAL PROPERTIES OF PASCAL FUNCTIONS

Theorem 1. Consider the proper sequence of Pascal functions $\{p_n(x,y)\}_{n=0}^{\infty} \in \Pi_k$ satisfying

$$p_{n+1}(x,y) = axp_n(x,y) + ayp_{n-k}(x,y), \qquad n \geqslant k, \tag{3}$$

with

$$p_0(x,y) = 0, \quad p_1(x,y) = a, \quad p_2(x,y) = a^2x, \quad \cdots, \quad p_k(x,y) = a^kx^{k-1}.$$

Then

$$\frac{\partial p_n(x,y)}{\partial x} = \frac{\partial p_{n+k}(x,y)}{\partial y} = \sum_{k=0}^{n} p_k(x,y)p_{n-k}(x,y). \tag{4}$$

Proof. One can establish the first part of (4) by induction. It is clear from (3) that

$$\frac{\partial p_{n+1}(x,y)}{\partial x} = ax\,\frac{\partial p_n(x,y)}{\partial x} + ap_n(x,y) + ay\,\frac{\partial p_{n-k}(x,y)}{\partial x} \tag{5}$$

and

$$\frac{\partial p_{n+k+1}(x,y)}{\partial y} = ax\,\frac{\partial p_{n+k}(x,y)}{\partial y} + ap_n(x,y) + ay\,\frac{\partial p_n(x,y)}{\partial y}. \tag{6}$$

The form of (5) and (6) together with the fact that the first part of (4) holds for $n = 1, 2, 3, \cdots, k$, proves it by induction. We now want to show

$$\frac{\partial p_n(x,y)}{\partial x} = \sum_{k=0}^{n} p_k(x,y)p_{n-k}(x,y). \tag{7}$$

Consider the generating function

$$G(t) = \sum_{n=0}^{\infty} p_n(x,y)t^n = \frac{at}{1 - axt - ayt^{k+1}}.$$

We have

$$\sum_{n=0}^{\infty} \frac{\partial p_n(x,y)}{\partial x}\,t^n = \frac{\partial G(t)}{\partial x} = \frac{a^2t^2}{(1 - axt - ayt^{k+1})^2} = [G(t)]^2.$$

This proves (7) and so we have established Theorem 1.

Corollary. For the Fibonacci polynomials defined before,

$$\frac{\partial F_n(x,y)}{\partial x} = \frac{\partial F_{n+1}(x,y)}{\partial y} = \sum_{k=0}^{n} F_k(x,y)F_{n-k}(x,y).$$

Proof. The corollary follows by taking $k = 1$ in Theorem 1.

Theorem 2. If

$$\frac{\partial p_n(x,y)}{\partial x} = p_{n,1}(x,y),$$

then define

$$p_{n,r}(x,y) = \sum_{k=0}^{n} p_{k,r-1}(x,y)P_{n-k}(x,y). \tag{8}$$

Now

$$p_{n,r}(x,y) = \frac{1}{r!}\frac{\partial^r p_n(x,y)}{\partial x^r}.$$

Proof. Differentiate the generating function $G(t)$ in the proof of Theorem 1, r times. Theorem 2 follows.

Theorem 3. If a proper sequence of Pascal functions

$$\left\{p_n(x,y)\right\}_{n=0}^{\infty} \in \Pi_k$$

satisfy (4), then they satisfy (3). (Converse of Theorem 1.)

Proof. Consider the first $(k+1)$ members of the sequence

$$a_0, a_1, a_2x, a_3x^2, \cdots, a_kx^{k-1}.$$

Because of (4) we have

$$\frac{d}{dx}(a_0) = 2a_0a_1,$$

and $a_1 \neq 0$, which gives $a_0 = 0$.

Further,

$$\frac{d}{dx}(a_2x) = a_2 = a_1^2.$$

Similarly, one may show

$$a_r = a_1^r = a^r, \qquad r = 1, 2, \cdots, k.$$

Now assume that (3) holds for $n = 0, 1, 2, 3, \cdots, m$. Let now

$$p^*_{m+1}(x,y) = \sum_{k=1}^{m} p_k(x,y)p_{m-k+1}(x,y).$$

Clearly, by Lemmas 1 and 2, we have $p^*_{m+1}(x,y) \in \Pi_k$.

Now, denote

$$p^{**}_{m+1}(x,y) = axp_m(x,y) + ayp_{m-k}(x,y).$$

We have because of Theorem 1

$$\frac{\partial p^{**}_{m+1}(x,y)}{\partial x} = p^*_{m+1}(x,y).$$

But we know, because $p_0(x,y) = 0$,

$$\frac{\partial p_{m+1}(x,y)}{\partial x} = p^*_{m+1}(x,y)$$

and this gives

$$p_{m+1}(x,y) = p^{**}_{m+1}(x,y)$$

by (1) and by Lemma 3. This proves that (3) holds, by mathematical induction. Hence we get Theorem 3.

3. PASCAL FUNCTIONS WHICH CAN BE PASCALISED

We now shift our attention to Pascal functions which can be "pascalised." Given a proper sequence of Pascal polynomials

$$\left\{ p_n(x,y) \right\}_{n=0}^{\infty} \in \Pi_k ,$$

form the associated array $\left\{ a_{ij} \right\} = A$. Now take

$$q_n(x,y) = \frac{\partial p_n(x,y)}{\partial x}$$

to get a new proper sequence in Π_k. Let $\left\{ b_{ij} \right\} = B$ be the associated Pascal array to this sequence. If we have the relation

$$b_{ij} = a_{ij} \binom{i+1}{1} \tag{9}$$

we say $\left\{ p_n(x,y) \right\}$ can be "pascalised" to the first order. If

$$q_n(x,y) = \frac{\partial^r p_n(x,y)}{\partial x^r}$$

and

$$b_{ij} = a_{ij} \binom{i+r}{r} r! \tag{10}$$

we say that the sequence $\left\{ p_n(x,y) \right\}$ can be pascalised to the r^{th} order.

Theorem 4. A necessary and sufficient condition that a proper sequence of $(k-1)^{st}$ order Pascal functions $\left\{ p_n(x,y) \right\}_{n=0}^{\infty}$ can be pascalised to the first order is that

$$p_n(x,y) = \sum_{j=0}^{[n/k]} a_j \binom{n-(k-1)j-1}{j} x^{n-kj-1} y^j \tag{11}$$

for some sequence of constants a_j.

Proof. We will first prove the theorem for the case $k = 2$. Consider the sequence $\left\{ p_n(x,y) \right\}_{n=0}^{\infty}$, and assume that the identity holds for $n = 0, 1, 2, \cdots, m$. We have then

$$p_m(x,y) = \sum_{j=0}^{[m/2]} a_j \binom{m-j-1}{j} x^{m-2j-1} y^j . \tag{12}$$

Now let

$$p_{m+1}(x,y) = \sum_{j=0}^{[(m+1)/2]} a_{j,m} \binom{m-j}{j} x^{m-2j} y^j$$

which gives

$$\frac{\partial p_{m+1}(x,y)}{\partial x} = \sum_{j=0}^{[m/2]} a_{j,m} \binom{m-j}{j} x^{m-2j-1} y^j (m-2j) . \tag{13}$$

Now comparing coefficients in (12) and (13) and using (9) we get

$$\binom{m-j}{j} (m-2j) a_{j,m} = \binom{m-j-1}{j} a_j$$

which gives

$$a_{j,m} = a_j$$

establishing part of the theorem for $k = 2$. The converse can be proved by retracing the steps.

Now, once the theorem is proved for the first order ($k = 2$), it holds for any $k \geqslant 1$, for given a proper sequence of Pascal functions of $(k-1)^{st}$ order, we can find its associated sequence of first order. The Pascal arrays for the derivatives of these two sequences is the same since the operator $\partial/\partial x$ will operate independently in the expansion of $p_n(x,y)$ with respect to coefficients in the associated Pascal array. This completes the proof of the theorem.

Theorem 5. If a proper sequence of k^{th} order Pascal functions can be pascalised to the first order, then all their associated sequences can be pascalised to first order.

Proof. Given in the last paragraph of the proof of Theorem 4.

Theorem 6. If a proper sequence of k^{th} order Pascal functions can be pascalised to first order, they can be pascalised to any order.

Proof. By arguments similar to the above, it is enough if we prove it for $k = 1$. Furthermore, it is enough to prove the theorem for the special case $a_j = 1$ for differential operators are unaffected by constant multiples.

We know from Theorem 4 that the first-order proper sequence of Pascal functions which can be pascalised to first order can be put in the form

$$p_n(x,y) = \sum_{j=0}^{[n/2]} \binom{n-j-1}{j} x^{n-2j-1}y^j a_j .$$

Now, as mentioned, $a_j = 1$, so that $p_n(x,y) = F_n(x,y)$, the Fibonacci polynomials. We then have

$$\frac{1}{r!}\,\frac{\partial p_{n+r+1}(x,y)}{\partial x^r} = \frac{1}{r!}\sum_{j=0}^{[(n+r)/2]} \frac{\partial}{\partial x^r}\,\frac{\binom{n+r-j}{j} x^{n+r-2j}y^j}{r!}$$

$$= \sum_{n+r-2j\geqslant 0} \binom{n+r-2j}{r}\binom{n+r-j}{j} x^{n-2j}y^j$$

which resembles (9) proving our theorem for Fibonacci polynomials, and so for Pascal functions. We demonstrate our result with the following:

Pascal Array for $F_n(x,y)$

1					
1	1				
1	2	1			
1	3	3	1		
1	4	6	4	1	
...	...	...	...	...	...

Pascal Array for $\dfrac{F_n(x,y)}{x}$

1					
2	2				
3	6	3			
4	12	12	4		
5	20	30	20	5	
...	...	...	...	...	...

Note 1: (2,2) = 2(1,1); (3,6,3) = 3(1,2,1); (4,12,12,4) = 4(1,3,3,1); ⋯. Each row has a common factor.

Note 2: Theorem 4 also says that each column has a common factor a_j. In the above all the $a_j = 1$.

Note 3: The Pascal array for $[\partial F_n(x,y)]/\partial x$ is also the Pascal array for

$$\sum_{k=0}^{n} F_k(x,y)F_{n-k}(x,y)$$

for both are equal by Theorem 1.

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REFERENCE

1. V. E. Hoggatt, Jr., and Calvin T. Long, "Divisibility Properties of Generalized Fibonacci Polynomials," *The Fibonacci Quarterly,* Vol. 12, No. 3 (April 1974), pp. 113–120.

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