ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by

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Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman; 709 Solano Dr., S.E.; Albuquerque, New Mexico 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within four months of the publication date.

DEFINITIONS

The Fibonacci numbers $F_n$ and the Lucas numbers $L_n$ satisfy

$$ F_{n+2} = F_{n+1} + F_n, \quad F_0 = 0, \quad F_1 = 1 \quad \text{and} \quad L_{n+2} = L_{n+1} + L_n, \quad L_0 = 2, \quad L_1 = 1. $$

Also $a$ and $b$ designate the roots $(1 + \sqrt{5})/2$ and $(1 - \sqrt{5})/2$, respectively, of $x^2 - x - 1 = 0$.

PROBLEMS PROPOSED IN THIS ISSUE

B-340 Proposed by Phil Mana, Albuquerque, New Mexico.

Characterize a sequence whose first 28 terms are:

- 1779, 1784, 1790, 1802, 1813, 1819, 1824, 1830, 1841, 1847, 1852, 1858, 1869, 1875,

B-341 Proposed by Peter A. Lindstrom, Genesee Community College, Batavia, New York.

Prove that the product $F_{2n}F_{2n+2}F_{2n+4}$ of three consecutive Fibonacci numbers with even subscripts is the product of three consecutive integers.

B-342 Proposed by Gregory Wuleczyn, Bucknell University, Lewisburg, Pennsylvania.

Prove that $2L_{n+3} + L_n^3 + 6L_{n+1}L_{n-1}$ is a perfect cube for $n = 1, 2, \ldots$.

B-343 Proposed by Verner E. Hoggatt, Jr., San Jose State University, San Jose, California.

Establish a simple expression for

$$ \sum_{k=1}^{n} \left( F_{2k-1}F_{2(n-k)+1} - F_{2k}F_{2(n-k+1)} \right). $$

B-344 Proposed by Frank Higgins, Naperville, Illinois.

Let $c$ and $d$ be real numbers. Find $\lim_{n \to \infty} x_n$, where $x_n$ is defined by $x_1 = c$, $x_2 = d$, and

$$ x_{n+2} = (x_{n+1} + x_n)/2 \quad \text{for} \quad n = 1, 2, 3, \ldots. $$

B-345 Proposed by Frank Higgins, Naperville, Illinois.

Let $r > s > 0$. Find $\lim_{n \to \infty} P_n$, where $P_n$ is defined by $P_1 = r + s$ and $P_{n+1} = r + s - (rs/P_n)$ for $n = 1, 2, 3, \ldots$.

SOLUTIONS

A FIBONACCI ALPHAMETIC

B-316 Proposed by J. A. H. Hunter, Fun with Figures, Toronto, Ontario, Canada.

Solve the alphametic

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Believe it or not, there must be no 8 in this!

Solution by Charles W. Trigg, San Diego, California.

$T < 5$, and no letter represents 8. There are five cases to consider.

1. If $2T + 1 = E$, and $T = 1$, then $E = 3$, and $O = 5$.
   - If $H = 6$, then $W = 9$, $I = 2$, and $2 + 2R = G$, impossible.
   - If $H = 7$, then $W = 0$, $I = 4$, and $I + 2R = G$, impossible.
   - If $H = 9$, then $I = 8$, which is prohibited.

2. If $2T + 1 = E$ and $T = 3$, then $E = 7$ and $O = 9$.
   - If $H = 4$, then $W = 8$, prohibited.
   - If $H = 5$, then $W = 9 = 0$.
   - If $H = 6$, then $W = 0$, and $4 + 2R = G$, impossible.

3. If $2T + 1 = E$ and $T = 4$, then $E = 9$, $O = 6$, and $H = W$.

4. If $2T = E$ and $T = 1$, then $E = 2$ and $O = 7$.
   - If $H = 3$, then $W = 8$, which is prohibited.
   - If $H = 4$, then $W = 9$ and $I = 8$ or 9.

5. If $2T = E$ and $T = 3$, then $E = 6$ and $O = 1$.
   - If $H = 4$, then $W = 1 = 0$.
   - If $H = 2$, then $W = 9$ and $5 + 2R = G$ or $G + 10$.

Whereupon, $R = 0$, $G = 5$, and $I = 4$. Thus the unique reconstructed addition is

$$391 + 32066 + 32066 = 65423.$$  


**LUCAS DIVISOR**

B-317 Proposed by Herta T. Freitag, Roanoke, Virginia.

Prove that $L_{2n-1}$ is an exact divisor of $L_{4n-1} - 1$ for $n = 1, 2, …$.

Solution by Gerald Bergum, Brookings, South Dakota.

Using the Binet formula together with $a\beta = -1$ and $a + \beta = 1$ we have

\[ L_{2n-1} - 1 = (a^n + \beta^n)(a^{n-1} + \beta^{n-1}) = a^{4n-1} + \beta^{4n-1} + (a\beta)^{2n-1}(a + \beta) = L_{4n-1} - 1. \]


**FIBONACCI SQUARE**

B-318 Proposed by Herta T. Freitag, Roanoke, Virginia.

Prove that $F_{4n}^2 + 8F_{2n}(F_{2n} + F_{6n})$ is a perfect square for $n = 1, 2, …$.

Solution by George Berzsenyi, Lamar University, Beaumont, Texas.

Using well known identities (see, for example, $I_{11}$ and $I_{12}$ in Hoggatt’s Fibonacci and Lucas Numbers) one finds that

\[ F_{2n}^2 + 8F_{2n}(F_{2n} + F_{6n}) = F_{2n}^2 + 8F_{2n}(F_{4n}L_{2n}) = F_{2n}^2 + 8F_{4n}(F_{2n}L_{2n}) = F_{2n}^2 + 8F_{4n}^2 \]

\[ = 9F_{4n}^2 = (3F_{4n})^2. \]
RERUN

B-319 Prove or disprove:

\[
\frac{1}{L_1} + \frac{1}{L_6} + \cdots = \frac{1}{\sqrt{5}} \left( \frac{1}{F_2} - \frac{1}{F_6} + \frac{1}{F_{10}} - \cdots \right).
\]


Also solved by Paul S. Bruckman, Mike Hoffman, and the Proposer.

A SUM

B-320 Proposed by George Berzsenyi, Beaumont, Texas.

Evaluate the sum:

\[
\sum_{k=0}^{n} F_k F_{k+2m}.
\]

Solution by Gerald Bergum, Brookings, South Dakota.

Using induction it is easy to show that

\[
\sum_{k=0}^{2t} F_k F_{k+d} = F_{2t} F_{2t+d+1}.
\]

If \(n\) is even, we have,

\[
\sum_{k=0}^{n} F_k F_{k+2m} = F_n F_{n+2m+1}.
\]

If \(n\) is odd, we have

\[
\sum_{k=0}^{n} F_k F_{k+2m} = F_{n-1} F_n + F_n F_{n+2m} = F_{n+1} F_{n+2m}.
\]


A RELATED SUM


Evaluate the sum:

\[
\sum_{k=0}^{n} F_k F_{k+2m+1}.
\]

Solution by Gerald Bergum, Brookings, South Dakota.

Using induction it is easy to show that

\[
\sum_{k=0}^{2t} F_k F_{k+d} = F_{2t} F_{2t+d+1}.
\]

If \(n\) is even, we have
\[ \sum_{k=0}^{n} F_k F_{k+2m+1} = F_n F_{n+2m+2}. \]

If \( n \) is odd, we have
\[ \sum_{k=0}^{n} F_k F_{k+2m+1} = F_{n-1} F_{n+2m+1} + F_n F_{n+2m+1} = F_{n+1} F_{n+2m+1}. \]

Also solved by the solvers of B-320.

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**ADVANCED PROBLEMS AND SOLUTIONS**

Show that
\[ L(m,n) - 25F(m,n) = 8L_{m+n} F_{m+1} F_{n+1}. \]

Solution by the Proposer.

It follows from the Binet formulas
\[ F_m = \frac{\alpha^m - \beta^m}{\alpha - \beta}, \quad L_m = \alpha^m + \beta^m \]
that
\[ 5F_m F_n = L_{m+n} - (\alpha^m \beta^n + \alpha^n \beta^m), \]
so that
\[ 5F_{it+j} F_{m-i+\nu-j} = L_{m+n} - (\alpha^{it+j} \beta^{-i+n-j} + \alpha^{-i+n-j} \beta^{it+j}), \]
\[ 5F_{m+i+j} F_{m-i+n-j} = L_{m+n} - (\alpha^{i+n-j} \beta^{-i+n-j} + \alpha^{-i+n-j} \beta^{i+n-j}). \]

Hence
\[ 25F_{it+j} F_{m-i+n-j} F_{m-i+n-j} = L_{m+n}^2 - L_{m+n} (\alpha^{i+n-j} \beta^{-i+n-j} + \alpha^{-i+n-j} \beta^{i+n-j}) + (\alpha^{2i+n-j} \beta^{-2i+n-j} + \alpha^{-2i+n-j} \beta^{2i+n-j}). \]

It follows that
\[ 25F(m,n) = (m+1)(n+1) L_{m+n}^2 + 4L_{m+n} F_{m+1} F_{n+1} + 2(-1)^m (n+1) F_{2m+2} + 2(-1)^n (m+1) F_{2n+2}. \]

Similarly,
\[ L(m,n) = (m+1)(n+1) L_{m+n}^2 + 4L_{m+n} F_{m+1} F_{n+1} + 2(-1)^m (n+1) F_{2m+2} + 2(-1)^n (m+1) F_{2n+2}. \]

Therefore,
\[ L(m,n) - 25F(m,n) = 8L_{m+n} F_{m+1} F_{n+1}. \]

Also solved by P. Bruckman.

**EDITORIAL REQUEST!** Send in your problem proposals!