ON SOME INVERSE TANGENT SUMMATIONS

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In this note, we derive the sums of a number of infinite series, some apparently new, in a rather simple manner. It is a simple result that

$$\sum_{n=1}^{\infty} \tan^{-1} \frac{x}{n^2+n+x^2} = \sum_{n=1}^{\infty} \left\{ \tan^{-1} \frac{x}{n} - \tan^{-1} \frac{x}{n+1} \right\} = \tan^{-1} x \,. \tag{1}$$

More generally, we have

$$\sum_{n=0}^{\infty} \left\{ \tan^{-1} F(n) - \tan^{-1} F(n+1) \right\} = \sum_{n=0}^{\infty} \tan^{-1} \frac{F(n) - F(n+1)}{1+F(n)F(n+1)} = \tan^{-1} F(0) - \lim_{n\to\infty} \tan^{-1} F(n)$$

for arbitrary F. In particular, for $F(n) = (an+b)/(cn+d)$, we obtain

$$\tan^{-1} \frac{bc-ad}{ab+cd} = \sum_{n=0}^{\infty} \tan^{-1} \frac{bc-ad}{n^2+An+B} \,, \tag{2}$$

where

$$A = 2(ab+cd)+1, \qquad B = b^2+d^2+ab+cd, \qquad a^2+c^2 = 1.$$

If in (2), we let $bc - ad = x$, $ab + cd = y$, then $b^2 + d^2 = x^2 + y^2$ giving

$$\tan^{-1} \frac{x}{y} = \sum_{n=0}^{\infty} \tan^{-1} \frac{x}{n^2+(2y+1)n+x^2+y^2+y} \,. \tag{3}$$

Then by differentiating (3) with respect to x and y, separately we obtain

$$\frac{y}{x^2+y^2} = \sum_{n=0}^{\infty} \frac{n^2+(2y+1)n+y^2+y-x^2}{[n^2+(2y+1)n+y^2+y+x^2]^2+x^2} \tag{4}$$

$$\frac{1}{x^2+y^2} = \sum_{n=0}^{\infty} \frac{(2n+2y+1)}{[n^2+(2y+1)n+y^2+y+x^2]^2+x^2} \,, \tag{5}$$

and also the following interesting special cases:

$$\tan^{-1} \frac{x}{x+1} = \sum_{n=1}^{\infty} \tan^{-1} \frac{x}{n^2+(2x+1)n+2x^2+x} \,, \tag{6}$$

$$\frac{1}{2} = \sum_{n=1}^{\infty} \frac{2n^2-1}{4n^4+1} \,, \tag{7}$$

(8)
$$\frac{1}{4} = \sum_{n=1}^{\infty} \frac{n}{4n^4+1},$$

(9)
$$\frac{1}{x^2+1} = \sum_{n=1}^{\infty} \frac{n^2+n-x^2}{(n^2+n+x^2)^2+x^2},$$

(10)
$$\frac{1}{x^2+1} = \sum_{n=1}^{\infty} \frac{2n+1}{(n^2+n+x^2)^2+x^2}.$$

To obtain analogous alternating sums, we let

$$F(n) = (-1)^n \left\{ \tan^{-1} \frac{an+b}{cn+d} - \tan^{-1} \frac{a}{c} \right\}$$

which leads to

(11)
$$\tan^{-1} \frac{x}{y} = \sum_{n=0}^{\infty} (-1)^n \tan^{-1} \frac{x(2n+2y+1)}{n^2+(2y+1)n+y^2+y-x^2}$$

and then by differentiating to

(12)
$$\frac{y}{x^2+y^2} = \sum_{n=0}^{\infty} \frac{(-1)^n \left\{n^2+(2y+1)n+y^2+y+x^2\right\}(2n+2y+1)}{\left\{n^2+(2y+1)n+y^2+y-x^2\right\}^2+(2n+2y+1)x^2},$$

(13)
$$\frac{1}{2(x^2+y^2)} = \sum_{n=0}^{\infty} \frac{(-1)^n \left\{n^2+(2y+1)n+y^2+y+x^2\right\}}{\left\{n^2+(2y+1)n+y^2+y-x^2\right\}^2+(2n+2y+1)x^2}.$$

These three latter formulae include the following special cases:

(14)
$$\pi - \tan^{-1} x = \sum_{n=1}^{\infty} (-1)^{n+1} \tan^{-1} \frac{(2n+1)x}{n^2+n-x^2},$$

(15)
$$\tan^{-1} x = \sum_{n=1}^{\infty} (-1)^{n+1} \tan^{-1} \frac{4nx}{4n^2-x^2-1},$$

(16)
$$\tanh^{-1} \frac{1}{y} = \sum_{n=0}^{\infty} (-1)^n \tanh^{-1} \frac{1}{2n+y+1},$$

(17)
$$\frac{\pi}{2} = \sum_{n=0}^{\infty} (-1)^n \tan^{-1} \frac{2n+3}{n^2+3n+1},$$

(18)
$$\frac{1}{8} = \sum_{n=1}^{\infty} \frac{(-1)^n n^3}{8n^4-4n^2+1} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^n n^2}{8n^4-4n^2+1},$$

(19)
$$\frac{1}{2x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n (n^2+n+x^2)}{(n^2+n-x^2)^2+(2n+1)^2x^2}.$$

To obtain a class of sums complementary to (2), we use another simple general method. Consider any product (finite or infinite)

$$P = \prod_n (a_n + ib_n), \qquad (a_n, b_n - \text{real}).$$

Then,

$$\arg P = \sum_n \tan^{-1} \frac{b_n}{a_n}, \tag{20}$$

$$|P|^2 = \prod_n (a_n^2 + b_n^2). \tag{21}$$

Applying (20) and (21) to the infinite products

$$\sin \pi z = \pi z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right), \qquad \cos \pi z = \prod_{k=1}^{\infty} \left(1 - \frac{4z^2}{(2k-1)^2}\right)$$

$$\frac{e^{\gamma a}\Gamma(z+1)}{\Gamma(z-a+1)} = \prod_{k=1}^{\infty} \left\{ e^{a/k} \left(1 - \frac{a}{x+k+iy}\right)\right\}, \quad J_0(xe^{3\pi i/4}) = \prod_{k=1}^{\infty} \left(1 + \frac{ix^2}{j_{0,k}^2}\right),$$

we obtain,

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{2xy}{k^2 - x^2 + y^2} = \tan^{-1} \frac{y}{x} - \tan^{-1} \frac{\tanh \pi y}{\tan \pi x}, \tag{22}$$

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{2xy}{(2k-1)^2 - x^2 + y^2} = \tan^{-1} \left\{ \tan \frac{\pi x}{2} \tanh \frac{\pi y}{2} \right\}, \tag{23}$$

$$\prod_{k=1}^{\infty} \left\{ 1 - \frac{2(x^2 - y^2)}{k^2} + \frac{(x^2 + y^2)^2}{k^4} \right\} = \frac{\sin^2 \pi x + \sinh^2 \pi y}{\pi(x^2 + y^2)} \tag{24}$$

$$\prod_{k=1}^{\infty} \left\{ 1 - \frac{8(x^2 - y^2)}{(2k-1)^2} + \frac{16(x^2 + y^2)^2}{(2k-1)^4} \right\} = \cos^2 \pi x + \sinh^2 \pi y \tag{25}$$

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{ay}{(x+k)^2 - a(x+k) + y^2} = \arg \Gamma(z+1)\Gamma(\bar{z} - a + 1) \tag{26}$$

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{x^2}{j_{0,k}^2} = \tan^{-1} \left\{ \frac{\mathrm{bei}(x)}{\mathrm{ber}(x)} \right\}, \tag{27}$$

$$\prod_{k=1}^{\infty} \left(1 + \frac{x^4}{j_{0,k}^4}\right) = \mathrm{ber}^2 x + \mathrm{bei}^2 x. \tag{28}$$

The right-hand side of (26) can be explicitly evaluated if either a or $\bar{z} + z - a$ is integral. If a is a positive integer,

$$\arg \Gamma(z+a)/\Gamma(z-a+1) = \sum_{k=0}^{a-1} \tan^{-1} \frac{y}{x-k}.$$

If $\bar{z} + z - a + 2 = m$ (positive integer), then $a = 2 + 2x - m$ and

$$\arg \Gamma(z+1)\Gamma(m-z-1) = \tan^{-1} \frac{\tanh \pi y}{\tan \pi x} - \sum_{k=2}^{m} \tan^{-1} \frac{y}{m-x-k}$$

(the last sum is to be taken as zero if $m = 1$). Further sums can be obtained by continued differentiation of all the previous sums containing at least one parameter.

Some of the sums given here appeared as problems in the Mathematical Tripos. A number of these are given among the exercises in Chapter XI of T. J. I'A. Bromwich, *An Introduction to the Theory of Infinite Series,* MacMillan, London, 1947.

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A NOTE ON THE GOLDEN ELLIPSE

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In [1], Huntley discusses some of the properties of the golden ellipse; that is, an ellipse whose ratio of the major axis to the minor axis is ϕ, the golden ratio. For example, Huntley shows that the eccentricity, e, of the golden ellipse is $1/\sqrt{\phi}$. This note is an examination of the golden ellipse as a conic section; see Fig. 1. It will be assumed that the plane does not pass through the vertex of the cone.

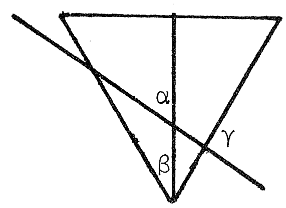

Figure 1

In [2] p. 355, it is shown that the eccentricity is determined by $\cos\alpha/\cos\beta = e$, where α and β are the angles in Fig. 1. Furthermore, for ellipses, $\beta < \alpha < 90°$.

In Fig. 1, the angle γ is formed by the intersection of the plane and the cone, in the plane through the axis of the cone and the main axis of the ellipse (easier seen than said). This angle will be referred to as the angle formed by the intersection of the plane and the cone.

Theorem. If α and β are such that $\sec\alpha = \phi$ and $\csc\beta = \phi$, then α and β are complementary, and the plane intersects the cone at a right angle, forming a golden ellipse. Conversely, if the plane intersects the cone at a right angle, forming a golden ellipse, then α and β are complementary, $\sec\alpha = \phi$, and $\csc\beta = \phi$.

Proof. Firstly, $\sin\beta = 1/\phi = \cos\alpha = \sin(\pi/2 - \alpha)$. Therefore, $\beta = \pi/2 - \alpha$. From Fig. 1 it follows that $\gamma = \pi - (\alpha+\beta)$. Hence, α and β are complementary and γ is a right angle.

Recalling that $\phi^2 - \phi - 1 = 0$,

$$\cos\beta = \sqrt{1-\sin^2\beta} = \sqrt{1 - 1/\sqrt{\phi^2}} = \sqrt{(\phi^2-1)/\phi^2} = \sqrt{\phi/\phi^2} = 1/\sqrt{\phi}\,.$$

Since $\cos\alpha = 1/\phi$, $e = \cos\alpha/\cos\beta = 1/\sqrt{\phi}$, and so the ellipse is golden.

Suppose that γ is a right angle and the ellipse is golden. Then, $\cos\alpha/\cos\beta = 1/\sqrt{\phi}$ and since

$$\pi/2 = \gamma = \pi - (\alpha+\beta),$$

α and β are complementary. Thus, $\cos\beta = \sin\alpha$. Now, $\sqrt{\phi}\cos\alpha = \cos\beta$ implies that

$$\phi\cos^2\alpha = \cos^2\beta = \sin^2\alpha = 1-\cos^2\alpha\,.$$

Therefore, $\cos^2\alpha = 1/(\phi+1) = 1/\phi^2$ and so $\sec\alpha = 1/\cos\alpha = \phi$. Also,

$$\csc\beta = 1/\sin\beta = 1/\cos\alpha = \phi\,.$$

REFERENCES

1. H. E. Huntley, "The Golden Ellipse," *The Fibonacci Quarterly,* Vol. 12, No. 1 (Feb. 1974), p. 38.
2. George B. Thomas, Jr., *Calculus and Analytic Geometry,* Addison-Wesley, Mass., 1968.

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