A DIRECT METHOD OF OBTAINING FAREY-FIBONACCI SEQUENCES

HANSRAJ GUPTA
Panjab University, Chandigarh, India

1. Krishnaswami Alladi [1], [2] has recently considered the problem of arranging in ascending order of magnitude the fractions F_j / F_k, $2 \leqslant j < k \leqslant n$ that can be obtained from the first n Fibonacci numbers by the relations

$$F_1 = F_2 = 1; \quad F_{m+1} = F_m + F_{m-1}, \quad m \geqslant 2$$

and discussed the symmetries and properties of this arrangement. As a consequence of these properties he gives a rapid method of constructing the Farey-Fibonacci sequence.

In this note we offer a direct method of obtaining such a Farey-Fibonacci sequence of fractions for $n \geqslant 3$. In fact once we prove the order of arrangement, the array on page 1 would give various properties with which Alladi started.

2. For n even, arrange the numbers from 2 to n in the order:

$$2 \quad 4 \quad 6 \quad \cdots \quad n \quad n-1 \quad n-3 \quad n-5 \quad \cdots \quad 3;$$

and for n odd, arrange them in the order:

$$3 \quad 5 \quad 7 \quad \cdots \quad n \quad n-1 \quad n-3 \quad n-5 \quad \cdots \quad 2.$$

The method is now best described with the help of an example. Let $n = 10$, then the numbers from 2 to 10 are written in the order

$$(1) \qquad 2 \quad 4 \quad 6 \quad 8 \quad 10 \quad 9 \quad 7 \quad 5 \quad 3.$$

With (1) as the base, complete the structure

$$\begin{array}{ccccccccc}
 & & & & 2 & & & & \\
 & & & & 3 & 2 & & & \\
 & & & 2 & 4 & 3 & & & \\
 & & & 3 & 5 & 4 & 2 & & \\
 & & 2 & 4 & 6 & 5 & 3 & & \\
 & & 3 & 5 & 7 & 6 & 4 & 2 & \\
 & 2 & 4 & 6 & 8 & 7 & 5 & 3 & \\
 & 3 & 5 & 7 & 9 & 8 & 6 & 4 & 2 \\
\hline
2 & 4 & 6 & 8 & 10 & 9 & 7 & 5 & 3
\end{array}$$

The building plan of the structure is simple and readily understood. Each figure in the configuration stands for a suffix of F. Thus, 5 stands for F_5, so to say. The base is separated from the superstructure by a line. The figures above the line provide the numerators, those on the base the denominators. For any numerator the figure vertically below it on the base provides the denominator. Thus 5 of the sixth row will give the fraction F_5/F_8. We start reading the figures from the top. The even numbered rows are read from right to left, the odd numbered rows from left to right. In other words, 2 is regarded as the first entry in each row. The configuration now gives the Farey-Fibonacci sequence straight away. In our example, it is:

$$F_2/F_{10}, \quad F_2/F_9, \quad F_3/F_{10}, \quad F_2/F_8, \quad F_4/F_{10}, \quad F_3/F_9, \quad \cdots, \quad F_3/F_4 .$$

In our scheme, there is no loss of labour in extending the structure. Thus, for $n = 11$, we obtain

				2					
				3	2				
			2	4	3				
			3	5	4	2			
		2	4	6	5	3			
		3	5	7	6	4	2		
	2	4	6	8	7	5	3		
	3	5	7	9	8	6	4	2	
2	4	6	8	10	9	7	5	3	
3	5	7	9	11	10	8	6	4	2

3. To show that our scheme does give the fractions in ascending order of magnitude, we have just to prove that

(i) $$F_2/F_3 < F_4/F_5 < \cdots < F_5/F_6 < F_3/F_4\,;$$

the two terms at the point of change-over being

$$F_{n-1}/F_n,\quad F_{n-2}/F_{n-1} \qquad \text{or} \qquad F_{n-2}/F_{n-1},\quad F_{n-1}/F_n$$

according as n is odd or even.

(ii) If $F_j/F_{j+1} > F_k/F_{k+1}$ then $F_j/F_{j+h} > F_k/F_{k+h}$, for every $h > 2$, and

(iii) $$F_3/F_{k+2} < F_2/F_k, \qquad k \geqslant 3.$$

The proof of (iii) is straightforward and is left to the reader.

Proof of (i).

$$1/1,\quad F_2/F_3,\quad F_3/F_4,\quad \cdots,\quad F_{n-1}/F_n$$

are convergents of the simple continued fraction

$$C_n = \frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\cdots\frac{1}{1} \quad \text{(with } n-1 \text{ partial quotients)}.$$

The well known properties of even and odd convergents provide immediately the proof of (i).

Proof of (ii). We have

(2) $$F_{j+1}/F_j < F_{k+1}/F_k.$$

Adding 1 on both sides of the inequality, we get

(3) $$F_{j+2}/F_j < F_{k+2}/F_k.$$

From (2) and (3) by addition, we obtain

(4) $$F_{j+3}/F_j < F_{k+3}/F_k.$$

The process can be continued to establish (ii).

We leave it to the reader to suggest a rule for obtaining the Farey-Fibonacci sequence for $n = m + 1$ from that for $n = m$.

4. We conclude with a formula which gives the position of the fraction F_j/F_k in the Farey-Fibonacci sequence for a given n, $2 \leqslant j < k \leqslant n$.

First observe that there are in all $\tfrac{1}{2}(n-1)(n-2)$ fractions in the sequence. It is now easy to see that F_j/F_{j+1} is the t^{th} term in the sequence, where

$$t = \begin{cases} \tfrac{1}{2}\left\{(n-2)(n-3)+j\right\}, & \text{when } j \text{ is even,} \\ \tfrac{1}{2}\left\{(n-1)(n-2)-(j-3)\right\}, & \text{when } j \text{ is odd.} \end{cases}$$

All that we need note now is that the position of F_j/F_{j+h}, $2 \leqslant j \leqslant m$, in the sequence for $n = m + h$, $h \geqslant 2$, is the same as the position of F_j/F_{j+1} in the sequence for $n = m + 1$.

These results follow at once from our scheme.

EXAMPLES: F_6/F_7 is the 31st term in the sequence for $n = 10$;

F_7/F_8 is the 43rd term in the sequence for $n = 11$;

and F_4/F_9 has the same position in the sequence for $n = 11$, as F_4/F_5 has in that for $n = 7$. This means that F_4/F_9 is the 12th term in the sequence for $n = 11$.

REFERENCES

1. Krishnaswami Alladi, "A Farey Sequence of Fibonacci Numbers," *The Fibonacci Quarterly,* Vol. 13, No. 1 (Feb. 1975), pp. 1–10.
2. ________________, "A Rapid Method to Form Farey Fibonacci Fractions," *The Fibonacci Quarterly,* Vol. 13, No. 1 (Feb. 1975), pp. 31–32.

Mailing address: 402 Mumfordganj, Allahabad, 211002, India.

******

ON CONSECUTIVE PRIMITIVE ROOTS

M. G. MONZINGO
Southern Methodist University, Dallas, Texas 75275

The purpose of this note is to determine which positive integers have their primitive roots consecutive. Of course, if "consecutive primitive roots" is taken to include integers which have only one primitive root, then 2, 3, 4, and 6 would qualify with primitive roots 1, 2, 3, and 5, respectively. It will be shown that 5, with primitive roots 2 and 3, is the only positive integer which has its primitive roots (plural) consecutive. It is well known that the only positive integers m, greater than 4, which have primitive roots are of the form p^n or $2p^n$, $n \geqslant 1$, p an odd prime. Most of these can be eliminated by the first two theorems.

Theorem 1. If $m = 2p^n$ $(m > 6)$, $n \geqslant 1$, p an odd prime, then the primitive roots are not consecutive.

Proof. Primitive roots must have inverses, and, consequently, must be relatively prime to the modulus. With $m > 6$, there will be at least two primitive roots. Therefore, there are at least two odd primitive roots and no even primitive roots; they are not consecutive.

Theorem 2. If $m = p^n$, $n \geqslant 2$, p an odd prime, then the primitive roots are not consecutive.

Proof. For $n \geqslant 3$,

$$p < p^{n-2}(p-1)\phi(p-1) = \phi(\phi(p^n)).$$

This implies that multiples of p occur within a span less than $\phi(\phi(p^n))$. Now, multiples of p are not relatively prime to the modulus, and are, therefore, not primitive roots. Since there are $\phi(\phi(p^n))$ primitive roots, they cannot be consecutive. For n = 2, $\phi(\phi(p^2)) = (p-1)\phi(p-1)$. For $p > 3$, $\phi(p-1) \geqslant 2$, and so,

$$(p-1)\phi(p-1) \geqslant 2(p-1) = 2p-2 = p+p-2 > p.$$

The conclusion follows as in the case $n \geqslant 3$. For $m = 3^2$, the primitive roots are 2 and 5, and not consecutive.

Lemma. If p is an odd prime greater than 5 and not equal to 7, 11, 13, 19, 31, 43, 61, then $2\sqrt{p-1} \leqslant \phi(p-1)$.

Proof. The conclusion is equivalent to $4(p-1) \leqslant [\phi(p-1)]^2$. Let $p - 1 = 2^a p_1^{a_1} \cdots p_n^{a_n}$, and suppose that $4(p-1) > [\phi(p-1)]^2$. Then,

$$2^{a+2}p_1^{a_1} \cdots p_n^{a_n} > 2^{2(a-1)}p_1^{2(a_1-1)} \cdots p_n^{2(a_n-1)}(p_1-1)^2 \cdots (p_n-1)^2. \tag{1}$$

If $p - 1 = 2^a$, then (1) reduces to $2^{a+2} > 2^{2(a-1)}$. This implies that $16 > 2^a$, or $a < 4$. Thus, $p = 3$ or 5.

Otherwise, (1) reduces to

$$8 > 2^{a-1}p_1^{a_1-2}(p_1-1)^2 p_2^{a_2-2}(p_2-1)^2 \cdots p_n^{a_n-2}(p_n-1)^2. \tag{2}$$

[Continued on page 394.]