ON TRIBONACCI NUMBERS AND RELATED FUNCTIONS

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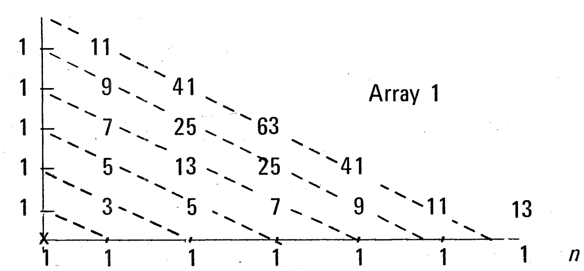
Stanton and Cowan [1] have discussed the two-dimensional analogue of Fibonacci Numbers. They dealt with numbers

$$g(n+1, r+1) = g(n+1, r) + g(n, r+1) + g(n, r)$$

$$g(n,0) = g(0,r) = 1 \quad r,n \geq 0 \text{ integers.}$$

Carlitz [2] has discussed in detail a more general form of $g(n,r)$. In this paper we get the Tribonacci Numbers from $g(n,r)$ and discuss properties of functions related to Tribonacci Numbers. Analogous identities have been established by Alladi [3] for Fibonacci Numbers. Bicknell and Hoggatt [4] have shown another method of getting Tribonacci Numbers.

The numbers $g(n,r)$ can be represented on a lattice as follows:



Array 1

The descending diagonals are denoted by dotted lines. The above figure is transformed into a Pascal-shaped triangle by changing the descending diagonals into rows

1				
1	1			
1	3	1		
1	5	5	1	
1	7	13	7	1
.	.	.	.	.

Array 2 Tribonacci Triangle

It is interesting to note that the sequence of diagonal sums in the Pascal-shaped triangle is

$$1, 1, 2, 4, 7, 13, 24, 44, 81, \cdots,$$

which is the Tribonacci sequence

$$T_n = T_{n-1} + T_{n-2} + T_{n-3}, \qquad T_0 = 0, \qquad T_1 = 1, \qquad T_2 = 1.$$

We now add variables (suitably) x^n, y^m on the arrays to make every row a homogeneous function in x and y.

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1				
x	y			
x^2	$3xy$	y^2		
x^3	$5x^2y$	$5xy^2$	y^3	
x^3	$7x^3y$	$13x^2y^2$	$7xy^3$	y^4

The rising diagonal sums give a sequence of functions T_n with the following rule of formation:

$$T_n(x,y) = xT_{n-1}(x,y) + yT_{n-2}(x,y) + xyT_{n-3}(x,y).$$

Let us denote the partial derivatives and convolutions by the following

$$t_n(x,y) = \frac{\partial T_n(x,y)}{\partial x}, \qquad t_n^*(x,y) = \frac{\partial T_n(x,y)}{\partial y}$$

$$\tau_n(x,y) = \sum_{k=0}^{n} T_k(x,y)T_{n-k}(x,y).$$

As in the case of Fibonacci Polynomials, do there exist relations between these functions? To get symmetric results we denote $\tau_n(x,y)$ by $\tau_{n+1}^*(x,y)$.

Theorem 1. $\tau_{n+1}^*(x,y) + y\tau_{n-1}^*(x,y) = t_n(x,y).$

Theorem 2. $\tau_n(x,y) + x\tau_{n-1}(x,y) = t_{n+1}^*(x,y)$

Theorem 3. $t_{n+1}^*(x,y) - t_n(x,y) = x\tau_{n-1}(x,y) - y\tau_{n-2}(x,y).$

Proofs. Theorem 3 follows immediately from Theorems 1 and 2. Since Theorem 2 is similar to Theorem 1 we prove only Theorem 1.

To prove Theorem 1 we would essentially have to show

(1) $$\tau_n(x,y) + y\tau_{n-2}(x,y) = t_n(x,y).$$

Assume that statement holds for $n = 0, 1, 2, 3, \cdots, m$. From the recurrence relation for $\tau_n(x,y)$ we see that

$$\frac{\partial T_{m+1}(x,y)}{\partial X} = T_m(x,y) + x\frac{\partial T_m(x,y)}{\partial X} + y\frac{\partial T_{m-1}(x,y)}{\partial X} + yT_{m-2}(x,y) + yx\frac{\partial T_{m-2}(x,y)}{\partial X}.$$

Now

$$\tau_{m+1}(x,y) + y\tau_{m-1}(x,y) = \sum_{k=0}^{m+1} T_k(x,y)T_{m-k+1}(x,y) + y\sum_{k=0}^{m-1} T_k(x,y)T_{m-k-1}(x,y)$$

$$= x\left[\sum_{k=0}^{m} T_k(x,y)T_{m-k}(x,y) + y\sum_{k=0}^{m-2} T_k(x,y)T_{m-k-2}(x,y)\right]$$

$$+ y\left[\sum_{k=0}^{m-1} T_k(x,y)T_{m-k-1}(x,y) + y\sum_{k=0}^{m-3} T_k(x,y)T_{m-k-3}(x,y)\right]$$

$$+ xy\left[\sum_{k=0}^{m-2} T_k(x,y)T_{m-k-2}(x,y) + y\sum_{k=0}^{m-4} T_k(x,y)T_{m-k-4}(x,y)\right] + g(x,y) + h(x,y)$$

applying recurrence for $T_n(x,y)$, where $g(x,y) + h(x,y)$ are the remainder terms from the first and second summations in each square bracket. Now using the recurrence we may simplify $g(x,y)$ and $h(x,y)$ to

$$g(x,y) = T_m(x,y), \qquad h(x,y) = yT_{m-2}(x,y)$$

which makes the right-hand side to be equal to the partial derivative

$$\frac{\partial T_{m+1}(x)}{\partial X}.$$

This means (1) holds for $n = m + 1$ and can be verified to hold for $n = 0, 1$. By mathematical induction it holds for all positive integral values of n.

We shall now discuss some more properties of the Tribonacci Triangle. If we attach the term x^m to every member of the $(m+1)^{th}$ row then the generating function of the $(n+1)^{th}$ column is

$$G_n(X) = \frac{X^n(1+X)^n}{(1-x)^{n+1}}$$

so that

$$\sum_{n=0}^{\infty} G_n(x) = \frac{1}{1-2x-x^2} \,. \tag{1}$$

Now (1) clearly indicates that the row sums of the Tribonacci Triangle are Pell-Numbers

$$P_n = 2P_{n-1} + P_{n-2}, \qquad P_0 = 0, \qquad P_1 = 1.$$

If on the other hand we shifted the $(n+1)^{th}$ column n steps downwards and in the new array added the term X^m to every member of the $(m+1)^{th}$ row, then the generating function of the $(n+1)^{th}$ column of this array would be

$$G_n^*(x) = \frac{X^{2n}(1+X)^n}{(1-X)^{n+1}}$$

so that

$$\sum_{n=0}^{\infty} G_n^*(X) = \frac{1}{1-x-x^2-x^3} \,. \tag{2}$$

Now (2) indicates that the rising diagonal sums of array (2) are Tribonacci Numbers. In fact if we attached X^mY^r to the $(m+1)^{th}$ for $(r+1)^{th}$ column element of the Tribonacci triangle we get the generating function of the $(n+1)^{th}$ column as

$$G_n(X,Y) = \frac{X^nY^n(1+x)^n}{(1-x)^{n+1}}$$

so that

$$\sum_{n=0}^{\infty} G_n(X,Y) = \frac{1}{1-X-XY-X^2Y} \tag{3}$$

which is the two-variable generating function of array (2). We conclude by considering the inverse of the following matrix.

$$\begin{pmatrix} 1 \\ 1 & 1 \\ 1 & 3 & 1 \\ 1 & 5 & 5 & 1 \\ 1 & 7 & 13 & 7 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}^{-1} = \begin{pmatrix} 1 \\ -1 & 1 \\ 2 & -3 & 1 \\ -6 & 10 & -5 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

Now denote by ${}^nT_r^*$ the $(n+1)^{th}$ row $(r+1)^{th}$ column element of

$$\begin{matrix} 1 \\ -1 & 1 \\ 2 & -3 & 1 \\ -6 & 10 & -5 & 1 \\ \cdot & \cdot & \cdot & \cdot\;\cdot \end{matrix}$$

Two interesting properties of ${}^nT_r^*$ stand out

P.1. $$\sum_{r=0}^{n} {}^nT_r^* = 0 \qquad n \geqslant 1 \qquad (= 1 \text{ for } n = 0)$$

P.2. $$\sum_{r=0}^{n} |^{n}T_r^*| = |^{n+1}T_0^*| \qquad (= 1 \text{ for } n = 0).$$

REMARKS. We wish to draw attention to the fact that we obtained Tribonacci Numbers from Stanton and Cowan's Diagram. Such a generalization to higher dimensions may be possible but it is very complicated as it is exceedingly difficult to picture these numbers. However there are other ways of obtaining these numbers as for example Tribonacci numbers from the expansion of $(1 + x + x^2)^n$ [4].

REFERENCES

1. R. G. Stanton and D. D. Gowan, "Note on a Square Functional Equation," *SIAM Rev.,* 12, 1970, pp. 277–279.
2. L. Carlitz, "Some q Analogues of Certain Combinatorial Numbers," *SIAM, Math. Anal.,* 1973, 4, pp. 433-446.
3. K. Alladi, "On Fibonacci Polynomials and Their Generalization," *The Fibonacci Quarterly,* to appear.
4. V. E. Hoggatt, Jr., and Marjorie Bicknell, "Generalized Fibonacci Polynomials," *The Fibonacci Quarterly,* Vol. 11, No. 5 (1973), p. 457.

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(6) $$\begin{cases} C_1R_1 + C_2R_2 + C_3R_3 = 1 \\ C_1R_1^2 + C_2R_2^2 + C_3R_3^2 = 3 \\ C_1R_1^3 + C_2R_2^3 + C_3R_3^3 = 6, \end{cases}$$

whose determinant is

(7) $$D = \begin{vmatrix} R_1 & R_2 & R_3 \\ R_1^2 & R_2^2 & R_3^2 \\ R_1^3 & R_2^3 & R_3^3 \end{vmatrix} = (R_1 - R_2)(R_1 - R_3)(R_2 - R_3) = 7.$$

Thus, using Cramer's rule, one obtains constants as

(8) $$\begin{cases} C_1 = \frac{1}{7} R_2R_3(R_3 - R_2)[6 - 3(R_2 + R_3) + R_2R_3] \\ C_2 = \frac{1}{7} R_1R_3(R_1 - R_3)[6 - 3(R_1 + R_3) + R_1R_3] \\ C_3 = \frac{1}{7} R_1R_2(R_2 - R_1)[6 - 3(R_1 + R_2) + R_1R_2], \end{cases}$$

which reduce simply to the fixed numbers

(9) $$C_1 = \frac{1}{7}(3 - R_3), \qquad C_2 = \frac{1}{7}(3 - R_1), \qquad C_3 = \frac{1}{7}(3 - R_2)$$

when many discovered relations between the three roots are taken into account. These involve the following.

Relations between the roots and the coefficient of the cubic gives

(10) $$R_1 + R_2 + R_3 = 2, \quad R_1R_2 + R_1R_3 + R_2R_3 = -1, \quad R_1R_2R_3 = -1,$$

while from the discriminant we have

(11) $$(R_1 - R_2)(R_1 - R_3)(R_2 - R_3) = \sqrt{49} = 7.$$

Use of these and the relation $R_1^2 + R_2^2 + R_3^2 = 6$ furnish, after some manipulation,

(12) $$\begin{cases} R_1R_3^2 + R_2R_1^2 + R_3R_2^2 = 4 \\ R_1R_2^2 + R_2R_3^2 + R_3R_1^2 = -3. \end{cases}$$

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