A FIBONACCI FORMULA OF LUCAS AND ITS SUBSEQUENT MANIFESTATIONS AND REDISCOVERIES

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Almost everyone who works with Fibonacci numbers knows that diagonal sums in the Pascal triangle give rise to the formula

(1) $$F_n = \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{n-k-1}{k}, \qquad n \geqslant 1,$$

but not many realize that

(2) $$F_{2n} = \sum_{k=0}^{\left[\frac{n-1}{2}\right]} (-1)^k \binom{n-k-1}{k} 3^{n-1-2k},$$

or that

(3) $$F_{3n} = 2 \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{n-k-1}{k} 4^{n-1-2k},$$

and that these are special cases of a very general formula given in 1878 by Edouard Lucas [5, Eqs. 74–76], [6, pp. 33–34].

As far as I can determine, formula (2) first appeared in our *Fibonacci Quarterly* as a problem posed by Lurline Squire [10] when she was studying number theory at West Virginia University. M. N. S. Swamy's solution invoked the use of Chebyshev polynomials. I was reminded of the formula recently when Leon Bernstein [1] found the formula again and asked me about it. He used a new technique involving algebraic number fields.

Formulas (2) and (3) generalize in a curious manner. On the one hand we have for *even* positive integers r

(4) $$\frac{F_{rn}}{F_r} = \sum_{k=0}^{\left[\frac{n-1}{2}\right]} (-1)^k \binom{n-k-1}{k} L_r^{n-1-2k}, \qquad 2|r,$$

but on the other hand for *odd* positive integers r we get the same terms but with all positive signs

(5) $$\frac{F_{rn}}{F_r} = \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{n-k-1}{k} L_r^{n-1-2k}, \qquad 2\nmid r,$$

where L_r is the usual Lucas number defined by $L_{n+1} = L_n + L_{n-1}$, with $L_0 = 2$, $L_1 = 1$, this of course in contrast with $F_{n+1} = F_n + F_{n-1}$ and $F_0 = 0$, $F_1 = 1$.

Formulas (4) and (5) may be written as a single formula in a clever way as noted by Hoggatt and Lind [4] who would write

(6) $$\frac{F_{rn}}{F_r} = \sum_{k=0}^{\left[\frac{n-1}{2}\right]} (-1)^{k(r-1)} \binom{n-k-1}{k} L_r^{n-1-2k},$$

valid now for any positive integers $n,r \geqslant 1$.

Formula (6) of Hoggatt and Lind was posed as a problem by James E. Desmond [11] and solved by him using a result of Joseph A. Raab [7]. The precise same problem was posed again by David Englund [12] and Douglas Lind pointed out that it was just the same formula.

Formulas (4) and (5) were obtained by Hoggatt and Lind [4] by calculations using compositions and generating functions. Although they cite Lucas [5] for a number of items they were evidently unaware that the formulas appear in Lucas in a far more general form. Since $L_r = F_{2r}/F_r$, formulas (4)–(5) can be written entirely in terms of F's.

Lucas introduced the general functions U, V defined by

$$U_n = \frac{a^n - b^n}{a - b}, \qquad V_n = a^n + b^n, \tag{7}$$

where a and b are the roots of the quadratic equation

$$x^2 - Px + Q = 0, \tag{8}$$

so that $a + b = P$ and $ab = Q$. When we have $x^2 - x - 1 = 0$, we get a and b as $(1+\sqrt{5})/2$ and $(1-\sqrt{5})/2$ and then $U_n = F_n$, $V_n = L_n$.

One of the general formulas Lucas gave is [6, pp. 33–34, note misprint in formula]

$$\frac{U_{rn}}{U_r} = \sum_{k=0}^{\left[\frac{n-1}{2}\right]} (-1)^k \binom{n-k-1}{k} V_r^{n-1-2k} Q^{rk}, \tag{9}$$

which unifies (4) and (5) and is more general than (6). Curiously, as we have intimated, Hoggatt and Lind do not cite this general formula.

Now of course, there are many other such formulas in Lucas' work. Two special cases should be paraded here for comparison. These are

$$L_{rn} = \sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k \frac{n}{n-k} \binom{n-k}{k} L_r^{n-2k} \quad \text{for even } r, \tag{10}$$

and

$$L_{rn} = \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{n}{n-k} \binom{n-k}{k} L_r^{n-2k} \quad \text{for odd } r. \tag{11}$$

These can be united in the same manner as (4)–(5) in (6). Thus

$$L_{rn} = \sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^{k(r-1)} \frac{n}{n-k} \binom{n-k}{k} L_r^{n-2k}. \tag{11.1}$$

There is nothing really mysterious about why such formulas exist. There are perfectly good formulas for the sums of powers of roots of algebraic equations tracing back to Lagrange and earlier. The two types of formulas we are discussing arise because of

$$\sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k \binom{n-k}{k} (xy)^k (x+y)^{n-2k} = \frac{x^{n+1} - y^{n+1}}{x - y}, \tag{12}$$

formula (1.60) in [3], and

(13) $$\sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k \frac{n}{n-k} \binom{n-k}{k} (xy)^k (x+y)^{n-2k} = x^n + y^n ,$$

formula (1.64) in [3], familiar formulas that say the same thing Lucas was saying. The reason it is not mysterious that (2) holds true, e.g., is that F_{2n} satisfies the second-order recurrence relation

$$F_{2n+2} = 3F_{2n} - F_{2n-2}$$

with which we associate the characteristic quadratic equation

$$x^2 = 3x - 1$$

so that a formula like (2) must be true. For formula (4) with $r = 4$ we note that $F_{4n+4} = 7F_{4n} - F_{4n-4}$. In general in fact,

(14) $$F_{rn+r} = L_rF_{rn} - F_{rn-r} \text{ for } \textit{even } r, \text{ or } F_{m+r} + F_{m-r} = L_rF_m ,$$

and

(15) $$F_{rn+r} = L_rF_{rn} + F_{rn-r} \text{ for } \textit{odd } r, \text{ or } F_{m+r} - F_{m-r} = L_rF_m .$$

Regularly spaced terms in a recurrent sequence of order two themselves satisfy such a recurrence. Set $u_n = F_{rn}$ to see this for then we have

(16) $$u_{n+1} = L_ru_n \pm u_{n-1}, \quad \text{with} \quad z^2 = L_rz \pm 1,$$

so we expect *a priori* that u_n must satisfy a formula rather like (1). Formulas like (12)–(13) give the sums of powers of the roots of the characteristic equation, whence the general formulas.

Formula (12) corresponds to (B.1) and (13) corresponds to (A.1) in Draim's paper [2] which the reader may also consult.

Another interesting fact is that these formulas are related to the Fibonacci polynomials introduced in a problem [9] and discussed at length by Hoggatt and others in later issues of the *Quarterly.* These are defined by

$$f_n(x) = xf_{n-1}(x) + f_{n-2}(x), \qquad n > 2,$$

with $f_1(x) = 1$ and $f_2(x) = x$.

In general

(17) $$f_n(x) = \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{n-k-1}{k} x^{n-2k-1} ,$$

whence for *odd r* we have by (5) that

(18) $$f_n(L_r) = \frac{F_{rn}}{F_r} .$$

Many other such relations can be deduced.

Finally we want to note two sets of inverse pairs given by Riordan [8] which he classifies as Chebyshev inverse pairs:

(19) $$f(n) = \sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k \frac{n}{n-k} \binom{n-k}{k} g(n-2k)$$

if and only if

(20) $$g(n) = \sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n}{k} f(n-2k);$$

and

$$f(n) = \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{n-2k+1}{n-k+1} \binom{n}{k} g(n-2k) \tag{21}$$

if and only if

$$g(n) = \sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k \binom{n-k}{k} f(n-2k). \tag{22}$$

Applying (19)–(20) to (10) we get the particularly nice formula

$$L_r^n = \sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n}{k} L_{r(n-2k)}, \quad r \text{ even}. \tag{23}$$

Using (21)–(22) on (4) we get the slightly more complicated formula

$$L_r^n = \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{n-2k+1}{n-k+1} \binom{n}{k} \frac{F_{r(n+1-2k)}}{F_r}. \tag{24}$$

I do not recall seeing (23) or (24) in any accessible location in our *Quarterly.*

If we let $r \to 0$ in (4) we can obtain the formula (1.72) in [3] of Lucas, which is also part of Desmond's problem [11] who does not cite Lucas,

$$n = \sum_{k=0}^{\left[\frac{n-1}{2}\right]} (-1)^k \binom{n-k-1}{k} 2^{n-2k-1}, \quad n \geqslant 1. \tag{25}$$

It is abundantly clear that the techniques we have discussed apply to many of the generalized sequences that have been introduced, e.g., Horadam's generalized Fibonacci sequence, but we shall not take the space to develop the obvious formulas. It is hoped that we have shed a little more light on a set of rather interesting formulas all due to Lucas.

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(iii) $$\left(\frac{p-1}{4}\right) G_n^2 + G_{n+1}^2 = G_{2n+1} \qquad (n \geq 1)$$

(iv) $$G_{n+2}^2 - \left(\frac{p-1}{4}\right)^2 G_n^2 = G_{2n+2} \qquad (n \geq 1)$$

(v) $$G_n = \sum_{r=0}^{n-1} \binom{n-1-r}{r} \left(\frac{p-1}{4}\right)^r \qquad (n \geq 1)$$

(vi) $$\left(\frac{p-1}{4}\right) \sum_{r=1}^{n} G_r = G_{n+2} - 1 \qquad (n \geq 1).$$

The proofs of the above results, which rely essentially on equations (2), (3) and (5), together with

$$\alpha - \beta = \sqrt{p}, \qquad \alpha + \beta = 1 \quad \text{and} \quad \alpha\beta = -\left(\frac{p-1}{4}\right),$$

are fairly straightforward and left to the reader. Of course, results such as these are not new. For example, (ii) was proved in a slightly more general form by E. Lucas as early as 1876 (see [1] page 396).

Finally, turning to the *vertical* sequences in the table given earlier, it follows from (v) that the sequence under G_n $(n \geq 1)$ is given by

(6) $$\left\{ \sum_{r=0}^{n-1} \binom{n-1-r}{r} (k-1)^r \right\} \qquad (k \geq 1),$$

so that for example the sequences under G_4 and G_5 are $\{2k-1\}$ and $\{k^2+k-1\}$, respectively. Alternatively, instead of using (6), we can apply the Binomial Theorem to (2) and obtain the general vertical sequence in the form

$$\left\{ \frac{1}{2^{n-1}} \sum_{\substack{r=1 \\ r \text{ odd}}}^{n} \binom{n}{r} (4k-3)^{(r-1)/2} \right\} \qquad (k \geq 1).$$

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