SUMS OF FIBONACCI RECIPROCALS

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Good [1] has shown that

$$(1)\qquad \sum_{m=0}^{n} \frac{1}{F_b} = 3 - \frac{F_{B-1}}{F_B}, \qquad n \geqslant 1,$$

where $b = 2^m$ and $B = 2^n$. (We use this notation to achieve clarity in printing.) A generalization may be given as follows:

$$(2)\qquad \sum_{m=0}^{n} \frac{1}{F_{kb}} = C_k - \frac{F_{kB-1}}{F_{kB}}, \qquad n,k \geqslant 1,$$

where C_k is independent of n and in fact

$$(3)\qquad C_k = \begin{cases} (1+F_{k-1})/F_k & \text{for even } k, \\ (1+F_{k-1})/F_k + 2/F_{2k} & \text{for odd } k. \end{cases}$$

For k = 1, 2, 3, ⋯, the first ten values of C_k are: 3, 2, 10/8, 1, 46/55, 3/4, 263/377, 2/3, 1674/2584, 7/11, ⋯. If we write C_k with denominator as F_{2k} then the numerators form the interesting sequence

3, 6, 10, 21, 46, 108, 263, 658, 1674, 4305, 11146, 28980, ⋯.

Formula (2) is easily proved by induction. Assuming it holds for n, then for $n+1$ we find that we have to show that

$$\frac{1}{F_{2kB}} = \frac{F_{kB-1}}{F_{kB}} - \frac{F_{2kB-1}}{F_{2kB}},$$

and this comes by setting $j = kB$ in the formula

$$(-1)^j F_j = F_{2j}F_{j-1} - F_{2j-1}F_j,$$

which may be proved directly by the Binet formula, or can be seen as a special case of the well known formula

$$F_{m+1}F_j + F_m F_{j-1} = F_{m+j}$$

when $m = -2j$ and using $F_{-j} = (-1)^{j+1}F_j$.

This shows that Formula (2) holds with C_k independent of n. Taking $n = 1$ we may determine C_k from

$$1/F_k + 1/F_{2k} = C_k - (F_{2k-1}/F_{2k}).$$

It is from this that we have found (3).

Since $F_j/F_{j-1} \to (1+\sqrt{5})/2$ as $j \to \infty$, we have a corollary

$$(4)\qquad \sum_{m=0}^{\infty} \frac{1}{F_{kb}} = C_k - \frac{1}{a}, \quad \text{with} \quad a = \frac{1+\sqrt{5}}{2}.$$

Our formula has an interesting application to sums of reciprocals of Fibonacci numbers in another way. As k and m take on all integer values such that $k \geqslant 0$ and $m \geqslant 0$, then $(2k+1)2^m$ generates each natural number once. Hence for absolutely convergent series we have the general transformation formula

(5)
$$\sum_{n=1}^{\infty} f(n) = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} f((2k+1)2^m) = \sum_{\substack{k=1\\ k\text{ odd}}}^{\infty}\sum_{m=0}^{\infty} f(k2^m).$$

Applying this to the Fibonacci numbers we have

(6)
$$\sum_{n=1}^{\infty}\frac{1}{F_n} = \sum_{\substack{k=1\\ k\text{ odd}}}^{\infty}\sum_{m=0}^{\infty}\frac{1}{F_{kb}} = \sum_{\substack{k=1\\ k\text{ odd}}}^{\infty}\left\{C_k - \frac{1}{a}\right\},\quad \text{by (4)}.$$

$$= 2.382\cdots + 0.632\cdots + 0.218\cdots + 0.080\cdots + 0.030\cdots$$
$$= 3.35988\cdots$$

as given by Brousseau [2, p. 45].

By some simple manipulations with the Binet formula $F_n = (a^n - (-1/a)^n)/\sqrt{5}$, formula (6) may be transformed into some variant forms that we believe are of interest. It is easy to verify the following:

(7)
$$\frac{F_{k-1}}{F_k} - \frac{1}{a} = -\frac{\sqrt{5}}{a^{2k}+1};$$

(8)
$$\frac{1}{F_{2k}} - \frac{\sqrt{5}}{a^{2k}+1} = \frac{\sqrt{5}}{a^{4k}-1};$$

(9)
$$\frac{1}{F_{2k}} + \frac{\sqrt{5}}{a^{4k}-1} = \frac{\sqrt{5}}{a^{2k}-1}.$$

To use these, we note that in view of (3), series (6) becomes

(10)
$$\sum_{n=1}^{\infty}\frac{1}{F_n} = \sum_{\substack{k=1\\ k\text{ odd}}}^{\infty}\left\{\frac{1}{F_k} + \frac{2}{F_{2k}} + \frac{F_{k-1}}{F_k} - \frac{1}{a}\right\},$$

so that by (7) we get

(11)
$$\sum_{n=1}^{\infty}\frac{1}{F_n} = \sum_{\substack{k=1\\ k\text{ odd}}}^{\infty}\left\{\frac{1}{F_k} + \frac{2}{F_{2k}} - \frac{\sqrt{5}}{a^{2k}+1}\right\}$$

$$= \sum_{\substack{k=1\\ k\text{ odd}}}^{\infty}\frac{1}{F_k} + 2\sum_{\substack{k=1\\ k\text{ odd}}}^{\infty}\frac{1}{F_{2k}} - \sqrt{5}\sum_{\substack{k=1\\ k\text{ odd}}}^{\infty}\frac{1}{a^{2k}+1}$$

$$= 1.8245\cdots + 2.2924\cdots - 0.7571\cdots.$$

Next, using (8), we get

(12)
$$\sum_{n=1}^{\infty}\frac{1}{F_n} = \sum_{\substack{k=1\\ k\text{ odd}}}^{\infty}\frac{1}{F_k} + \sum_{\substack{k=1\\ k\text{ odd}}}^{\infty}\frac{1}{F_{2k}} + \sqrt{5}\sum_{\substack{k=1\\ k\text{ odd}}}^{\infty}\frac{1}{a^{4k}-1}$$

$$= 1.8245\cdots + 1.1462\cdots + 0.389082\cdots.$$

Finally, using (9), this becomes

(13)
$$\sum_{n=1}^{\infty}\frac{1}{F_n} = \sum_{\substack{k=1\\ k\text{ odd}}}^{\infty}\frac{1}{F_k} + \sqrt{5}\sum_{\substack{k=1\\ k\text{ odd}}}^{\infty}\frac{1}{a^{2k}-1}$$

$$= 1.8245\cdots + 1.5353\cdots.$$

This last form of our result is most interesting because it is not at all what we get if we transform the reciprocals by simple bisection.

By bisection it is easy to see that

$$\sum_{n=1}^{\infty} \frac{1}{F_n} = \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{1}{F_k} + \sum_{\substack{j=1 \\ j \text{ even}}}^{\infty} \frac{1}{F_j} = \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{1}{F_k} + \sum_{n=1}^{\infty} \frac{\sqrt{5}}{a^{2n} - a^{-2n}},$$

whence

$$\sum_{n=1}^{\infty} \frac{1}{F_n} = \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{1}{F_k} + \sqrt{5} \sum_{n=1}^{\infty} \frac{a^{2n}}{a^{4n} - 1} \tag{14}$$

Comparing this with (13) we find the interesting equivalence

$$\sum_{n=1}^{\infty} \frac{a^{2n}}{a^{4n} - 1} = \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{1}{a^{2k} - 1}, \qquad a = \frac{1+\sqrt{5}}{2}. \tag{15}$$

The series on the right seems to converge twice as fast as that on the left, and six terms give the sum as $0.68663 \cdots$, whereas it takes 12 terms of the other series to get this.

Using the Binet formula it is also possible to rewrite (12) as

$$\sum_{n=1}^{\infty} \frac{1}{F_n} = \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} (a^{3k} + a^{2k} + a^k + 1 - 2a^k) \frac{\sqrt{5}}{a^{4k} - 1} = \sqrt{5} \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \left\{ \frac{1}{a^k - 1} - \frac{2a^k}{a^{4k} - 1} \right\} \tag{16}$$

$$= (2.083313 \cdots - 0.580727 \cdots)\sqrt{5} = (1.5025865492 \cdots)\sqrt{5} = 3.359885665 \cdots .$$

A preliminary form of this paper was written in October 1975 and communicated to H. W. Gould and I. J. Good later. The author is also indebted to H. W. Gould for suggestions leading to the presentation of the ideas in the present form. A generalization of the main results here will appear in another paper [3]. A generalization of formula (5) will appear in Gould [4]. See [5] for an earlier treatment.

REFERENCES

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