ON THE MULTINOMIAL THEOREM

DAVID LEE HILLIKER
The Cleveland State University, Cleveland, Ohio 44115

The Multinomial Expansion for the case of a nonnegative integral exponent n can be derived by an argument which involves the combinatorial significance of the multinomial coefficients. In the case of an arbitrary exponent n these combinatorial techniques break down. Here the derivation may be carried out by employment of the Binomial Theorem for an arbitrary exponent coupled with the Multinomial Theorem for a nonnegative integral exponent. See, for example, Chrystal [1] for these details. We have observed (Hilliker [6]) that in the case where n is not equal to a nonnegative integer, a version of the Multinomial Expansion may be derived by an iterative argument which makes no reference to the Multinomial Theorem for a nonnegative integral exponent. In this note we shall continue our sequence of expositions of the Binomial Theorem, the Multinomial Theorem, and various Multinomial Expansions (Hilliker [2], [3], [4], [5], [6], [7]) by making the observation that this iterative argument can be modified to cover the nonnegative integral case:

(1) $$\left(\sum_{i=1}^{r} a_i\right)^n = \sum_{n_1+n_2+\cdots+n_r=n} \binom{n}{n_1, n_2, \cdots, n_r} a_1^{n_1} a_2^{n_2} \cdots a_r^{n_r},$$

where $n_1, n_2, \cdots, n_r$ are nonnegative integers and where the multinomial coefficients are given by

$$\binom{n}{n_1, n_2, \cdots, n_r} = \frac{n!}{n_1!\, n_2! \cdots n_r!}.$$

As before (Hilliker [6]) we begin with a *triple summation expansion:*

(2) $$\left(\sum_{i=1}^{r} a_i\right)^n = \sum_{j=1}^{r}\sum_{k=1}^{n} \binom{n}{k} a_j^k \left(\sum_{\ell=1}^{j-1} a_\ell\right)^{n-k}.$$

Here, we are using the convention that the empty sum is zero and that $0^0 = 1$.

We next assert that *the Multinomial Theorem (1) is covered by the Formula (2).* To see this, let us make a change of notation and write Formula (2) as

(3) $$\left(\sum_{i=1}^{r} a_i\right)^n = \sum \binom{n}{\ell_2} a_{\ell_1}^{\ell_2} \left(\sum_{\ell=1}^{\ell_1-1} a_\ell\right)^{n-\ell_2},$$

where the double summation on the right is taken under ℓ_1 and ℓ_2 with $1 \leqslant \ell_1 \leqslant r$ and $1 \leqslant \ell_2 \leqslant n$. We single out the terms for which $n - \ell_2 = 0$ and write (3) as

(4) $$\left(\sum_{i=1}^{r} a_i\right)^n = \sum_{n-\ell_2>0} \binom{n}{\ell_2} a_{\ell_1}^{\ell_2} \left(\sum_{\ell=1}^{\ell_1-1} a_\ell\right)^{n-\ell_2} + \sum_{n-\ell_2=0} a_{\ell_1}^{\ell_2}.$$

Note that, for nonzero terms, $\ell_1 = 1$ implies that $n - \ell_2 = 0$, so that the range in the summation with $n - \ell_2 > 0$ is $2 \leqslant \ell_1 \leqslant r$ and $1 \leqslant \ell_2 \leqslant n - 1$.

We now apply Formula (3) to the summation under ℓ on the right side of (4). This iterative process may be continued. After m iterations of Formula (3), $m \geqslant 0$ and not too large, we obtain

$$(5)\qquad \left(\sum_{i=1}^{r} a_i\right)^n = \sum_{n-\ell_2-\cdots-\ell_{2m}>0} \binom{n}{\ell_2}\binom{n-\ell_2}{\ell_4}\cdots\binom{n-\ell_2-\cdots-\ell_{2m}}{\ell_{2m+2}}$$

$$\times\, a_{\ell_1}^{\ell_2} a_{\ell_3}^{\ell_4}\cdots a_{\ell_{2m+1}}^{\ell_{2m+2}} \left(\sum_{\ell=1}^{\ell_{2m+1}-1} a_\ell\right)^{n-\ell_2-\cdots-\ell_{2m+2}}$$

$$+\sum_{k=1}^{m}\ \sum_{n-\ell_2-\cdots-\ell_{2k}=0} \binom{n}{\ell_2}\binom{n-\ell_2}{\ell_4}\cdots\binom{n-\ell_2-\cdots-\ell_{2k-2}}{\ell_{2k}} a_{\ell_1}^{\ell_2} a_{\ell_3}^{\ell_4}\cdots a_{\ell_{2k-1}}^{\ell_{2k}}\,.$$

Here, the indices are subject to the restrictions

$$(6)\qquad \begin{cases} 1 \leqslant \ell_1 \leqslant r \\ 1 \leqslant \ell_{2i+1} \leqslant \ell_{2i-1}-1, & \text{for } 1 \leqslant i \leqslant m, \\ 1 \leqslant \ell_{2i+2} \leqslant n-\ell_2-\cdots-\ell_{2i}, & \text{for } 0 \leqslant i \leqslant m. \end{cases}$$

Formula (5) is meaningful as long as $m < r$, so that the first two inequalities in (6) are possible and as long as

$$(7)\qquad m < n,$$

so that the last inequality in (6) is possible. We let $m = r-1$. Then, by (6) we have $\ell_{2r-1} = 1$, for otherwise, we would have $\ell_1 > r$. Consequently, for nonzero terms,

$$n-\ell_2-\cdots-\ell_{2r} = 0.$$

Formula (5) now takes the form

$$(8)\qquad \left(\sum_{i=1}^{r} a_i\right)^n = \sum_{n-\ell_2-\cdots-\ell_{2r}=0} \binom{n}{\ell_2}\binom{n-\ell_2}{\ell_4}\cdots\binom{n-\ell_2-\cdots-\ell_{2r-2}}{\ell_{2r}} a_{\ell_1}^{\ell_2} a_{\ell_3}^{\ell_4}\cdots a_{\ell_{2r-1}}^{\ell_{2r}}$$

$$+\sum_{k=1}^{r-1}\ \sum_{n-\ell_2-\cdots-\ell_{2k}=0} \binom{n}{\ell_2}\binom{n-\ell_2}{\ell_4}\cdots\binom{n-\ell_2-\cdots-\ell_{2k-2}}{\ell_{2k}} a_{\ell_1}^{\ell_2} a_{\ell_3}^{\ell_4}\cdots a_{\ell_{2k-1}}^{\ell_{2k}}$$

$$= \sum_{k=1}^{r}\ \sum_{n-\ell_2-\cdots-\ell_{2k}=0} \binom{n}{\ell_2}\binom{n-\ell_2}{\ell_4}\cdots\binom{n-\ell_2-\cdots-\ell_{2k-2}}{\ell_{2k}} a_{\ell_1}^{\ell_2} a_{\ell_3}^{\ell_4}\cdots a_{\ell_{2k-1}}^{\ell_{2k}}\,.$$

If the range of ℓ_{2i}, for $1 \leqslant i \leqslant r$, is extended to include 0, then, the summation under k reduces to a single term, $k = r$; the restriction (7) may be lifted; and, by (6), the subscripts are uniquely determined: $\ell_1 = r$, $\ell_3 = r-1$, $\cdots$, $\ell_{2r-1} = 1$. The coefficients may be written as

$$\binom{n}{\ell_2}\binom{n-\ell_2}{\ell_4}\cdots\binom{n-\ell_2-\cdots-\ell_{2r-2}}{\ell_{2r}} = \frac{n(n-1)\cdots(n-\ell_2-\cdots-\ell_{2r}+1)}{\ell_2!\,\ell_4!\cdots\ell_{2r}!} = \frac{n!}{\ell_2!\,\ell_4!\cdots\ell_{2r}!}\,.$$

It now follows from (8) that

$$\left(\sum_{i=1}^{r} a_i\right)^n = \sum_{n-\ell_2-\cdots-\ell_{2r}=0} \frac{n!}{\ell_2!\,\ell_4!\cdots\ell_{2r}!}\, a_r^{\ell_2} a_{r-1}^{\ell_4}\cdots a_1^{\ell_{2r}}\,.$$

With a change of notation, the Multinomial Theorem (1) now follows.

REFERENCES

1. G. Chrystal, *Textbook of Algebra,* Vols. I and II, Chelsea, N.Y., 1964. This is a reprint of works of 1886 and 1889. Also reprinted by Dover, New York, 1961.
2. David Lee Hilliker, "A Study in the History of Analysis up to the Time of Leibniz and Newton in Regard to Newton's Discovery of the Binomial Theorem. First part, Contributions of Pascal," *The Mathematics Student,* Vol. XL, No. 1 (1972), 28–34.

3. David Lee Hilliker, "A Study in the History of Analysis up to the Time of Liebniz and Newton in regard to Newton's Discovery of the Binomial Theorem. Second part, Contributions of Archimedes," *The Mathematics Student,* Vol. XLII, No. 1 (1974), pp. 107–110.
4. David Lee Hilliker, "A Study in the History of Analysis up to the Time of Liebniz and Newton in Regard to Newton's Discovery of the Binomial Theorem. Third part, Contributions of Cavalieri," *The Mathematics Student,* Vol. XLII, No. 2 (1974), pp. 195–200.
5. David Lee Hilliker, "A Study in the History of Analysis up to the Time of Liebniz and Newton in Regard to Newton's Discovery of the Binomial Theorem. Fourth part, Contributions of Newton," *The Mathematics Student,* Vol. XLII, No. 4 (1974), pp. 397–404.
6. David Lee Hilliker, "On the Infinite Multinomial Expansion," *The Fibonacci Quarterly,* Vol. 15, No. 3, pp. 203–205.
7. David Lee Hilliker, "On the Infinite Multinomial Expansion, II," *The Fibonacci Quarterly,* Vol. 15, No. 5, pp. 392–394.

****

[Continued from page 21.]

Also, since α and β satisfy (4), we have the equations

$$\alpha^{n+2} = \alpha^{n+1} + \left(\frac{p-1}{4}\right)\alpha^n, \qquad \beta^{n+2} = \beta^{n+1} + \left(\frac{p-1}{4}\right)\beta^n \qquad (n \geq 1).$$

Therefore, using (3), it follows that

$$G_{n+2} = \frac{\alpha^{n+2}-\beta^{n+2}}{\sqrt{p}} = \frac{\alpha^{n+1} + \left(\frac{p-1}{4}\right)\alpha^n - \beta^{n+1} - \left(\frac{p-1}{4}\right)\beta^n}{\sqrt{p}}$$

$$= \frac{\alpha^{n+1}-\beta^{n+1}}{\sqrt{p}} + \left(\frac{p-1}{4}\right)\frac{\alpha^n-\beta^n}{\sqrt{p}} = G_{n+1} + \left(\frac{p-1}{4}\right) G_n .$$

Thanks to (5) it is now a simple matter (despite the complicated appearance of (2)) to generate terms of the sequence $\{G_n\}$, for any choice of p. Assuming that we are interested only in integer-valued sequences, (5) tells us to take p of the form $4k+1$; namely $p = 1, 5, 9, 13, 17, \cdots$. Thus the first five such sequences start as follows:

p	$\frac{p-1}{4}$	G_1	G_2	G_3	G_4	G_5	G_6	G_7	G_8	G_9	G_{10}	$\cdots$
1	0	1	1	1	1	1	1	1	1	1	1	$\cdots$
5	1	1	1	2	3	5	8	13	21	34	55	$\cdots$
9	2	1	1	3	5	11	21	43	85	171	341	$\cdots$
13	3	1	1	4	7	19	40	97	217	508	1159	$\cdots$
17	4	1	1	5	9	29	65	181	441	1165	2929	$\cdots$

We can use the above table to guess at various properties of the generalized Fibonacci sequence $\{G_n\}$, especially if our knowledge of $\{F_n\}$ is taken into account. Generalizations of some of the better-known properties of $\{F_n\}$ are listed below. Of course, in each case, the original result may be found by taking

$$p = 5, \qquad \frac{p-1}{4} = 1 \quad \text{and} \quad G_n = F_n .$$

(i)
$$\lim_{n\to\infty} \frac{G_{n+1}}{G_n} = \frac{1+\sqrt{p}}{2}$$

(ii)
$$G_n \cdot G_{n+2} - G_{n+1}^2 = (-1)^{n+1}\left(\frac{p-1}{4}\right)^n \qquad (n \geq 1)$$

[Continued on page 29.]