A FORMULA FOR $\sum_{1}^{n} F_k(x)y^{n-k}$ AND ITS GENERALIZATION TO r-BONACCI POLYNOMIALS

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1. INTRODUCTION

Some years ago, Carlitz [1] had asked the readers to show that

$$(1)\qquad \sum_{0}^{n-1} F_k 2^{n-k-1} = 2^n - F_{n+2}$$

and

$$(2)\qquad \sum_{0}^{n-1} L_k 2^{n-k-1} = 3(2^n) - L_{n+2},$$

where F_n and L_n are the n^{th} Fibonacci and Lucas numbers. Recently, King [2] generalized these results to obtain the expressions:

$$(3)\qquad \sum_{0}^{n-1} T_k y^{n-k-1} = \frac{(T_0 y + T_{-1})y^n - T_n y - T_{n-1}}{y^2 - y - 1}$$

and

$$(4)\qquad \sum_{0}^{n-1} T_k 2^{n-k-1} = T_2(2^n) - T_{n+2},$$

where the generalized Fibonacci numbers T_n are defined by

$$T_n = T_{n-1} + T_{n-2}, \qquad T_1 = a, \qquad T_2 = b.$$

The purpose of this article is to generalize these results to sums of the form $\Sigma F_k(x)y^{n-k}$, $\Sigma L_k(x)y^{n-k}$, $\Sigma H_k(x)y^{n-k}$, where $F_k(x)$, $L_k(x)$ and $H_k(x)$ are, respectively, Fibonacci, Lucas and generalized Fibonacci Polynomials, and then finally to extend these results to r-bonacci polynomials.

2. FIBONACCI AND LUCAS POLYNOMIALS AS COEFFICIENTS

The Fibonacci polynomials $F_n(x)$ are defined by [3]

$$(5)\qquad F_n(x) = xF_{n-1}(x) + F_{n-2}(x)$$

with $F_0(x) = 0$, $F_1(x) = 1$. Now consider the sum

$$S = \sum_{1}^{n} F_k(x)y^{n-k} = y^{n-1} + xy^{n-2} + \sum_{3}^{n} [xF_{k-1}(x) + F_{k-2}(x)]y^{n-k}$$

$$= y^{n-1} + xy^{n-2} + xy^{-1}\sum_{2}^{n-1} F_k y^{n-k} + y^{-1}\sum_{1}^{n-2} F_k y^{n-k}$$

$$= y^{n-1} + xy^{-1}\{S - F_n(x)\} + y^{-2}\{S - F_{n-1}(x)y - F_n(x)\}$$

Hence,

$$(y^2 - xy - 1)S = y^{n+1} - yF_{n+1}(x) - F_n(x).$$

Letting

(6) $$G_n(x,y) = y^{n+1} - yF_{n+1}(x) - F_n(x),$$

we may write S as

(7) $$S = \sum_{1}^{n} F_k(x)y^{n-k} = \frac{G_n(x,y)}{G_1(x,y)}, \qquad G_1(x,y) \neq 0.$$

The Lucas polynomials $L_n(x)$ are defined by [3]

(8) $$L_n(x) = xL_{n-1}(x) + L_{n-2}(x)$$

with $L_0(x) = 2,\ L_1(x) = x.$

It may be shown by induction or otherwise that

$$L_n(x) = F_{n+1}(x) + F_{n-1}(x).$$

Hence,

$$\sum_{1}^{n} L_k(x)y^{n-k} = \sum_{1}^{n} F_{k+1}(x)y^{n-k} + \sum_{1}^{n} F_{k-1}(x)y^{n-k} = \sum_{2}^{n+1} F_k(x)y^{n+1-k} + \sum_{0}^{n-1} F_k(x)y^{n-1-k}$$

$$= \sum_{1}^{n+1} F_k(x)y^{n+1-k} + \sum_{1}^{n-1} F_k(x)y^{n-1-k} - F_1(x)y^n = \frac{G_{n+1}(x,y) + G_{n-1}(x,y)}{G_1(x,y)} - y^n,$$

using (7)

$$= \frac{xy^{n+1} + 2y^n - y\,\{F_{n+2}(x) + F_n(x)\} - \{F_{n+1}(x) + F_{n-1}(x)\}}{G_1(x,y)}.$$

Therefore

(9) $$\sum_{1}^{n} L_k(x)y^{n-k} = \frac{xy^{n+1} + 2y^n - yL_{n+1}(x) - L_n(x)}{y^2 - xy - 1}.$$

By letting $x = 1,\ y = 2$ in results in (7) and (9), we obtain

(10) $$\sum_{1}^{n} F_k 2^{n-k} = 2^{n+1} - F_{n+3} = 2^n \cdot F_3 - F_{n+3}$$

and

(11) $$\sum_{1}^{n} L_k 2^{n-k} = 2^{n+2} - L_{n+3} = 2^n \cdot L_3 - L_{n+3}$$

which are the results of Carlitz [1]. Further, by letting $x = y = 2$ in (7) we get

(12) $$\sum_{1}^{n} P_k 2^{n-k} = P_{n+2} - 2^{n+1} = P_{n+2} - 2^n \cdot P_2,$$

where P_n is the n^{th} Pell number.

3. GENERALIZED FIBONACCI POLYNOMIALS AS COEFFICIENTS

Let us define the generalized Fibonacci polynomials $H_n(x)$ as

(13) $$H_n(x) = xH_{n-1}(x) + H_{n-2}(x)$$

with $H_0(x)$ and $H_1(x)$ arbitrary. It is obvious that the polynomials $F_n(x)$ are obtained by letting $H_0(x) = 0$, $H_1(x) = 1$, while the Lucas polynomials $L_n(x)$ are obtained by letting $H_0(x) = 2$ and $H_1(x) = 1$. In fact, it can be established that $H_n(x)$ is related to $F_n(x)$ by the relation

$$H_n(x) = H_1(x)F_n(x) + H_0(x)F_{n-1}(x).$$

Hence,

$$\sum_1^n H_k(x)y^{n-k} = H_1(x)\sum_1^n F_k(x)y^{n-k} + H_0(x)\sum_1^n F_{k-1}(x)y^{n-k}$$

$$= H_1(x)\frac{G_n(x,y)}{G_1(x,y)} + H_0(x)\sum_1^{n-1} F_k y^{n-1-k}, \text{ using (7)}$$

$$= \frac{H_1(x)G_n(x,y) + H_0(x)G_{n-1}(x,y)}{G_1(x,y)} .$$

The right-hand side may be simplified to show that

$$\sum_1^n H_k(x)y^{n-k} = \frac{H_1(x)y^{n+1} + H_0(x)y^n - yH_{n+1}(x) - H_n(x)}{y^2 - xy - 1} . \tag{14}$$

Some special cases of interest obtainable from (14) are,

$$\sum_1^n H_k(x)\cdot x^{n-k} = H_{n+2}(x) - x^n H_2(x), \qquad \sum_1^n H_k(x) = \frac{1}{x}[H_{n+1}(x) + H_n(x) - H_1(x) - H_0(x)] ,$$

$$\sum_1^n (-1)^{k+1}H_k(x) = \frac{1}{x}[(-1)^{n+1}\{H_{n+1}(x) - H_n(x)\} + \{H_1(x) - H_0(x)\}] .$$

It should be noted that by letting $x = 1$, $H_0(x) = a$ and $H_1(x) = b$ in (13), we generate the generalized Fibonacci numbers H_n defined earlier by Horadam [4]. From (14) it is seen that for these generalized Fibonacci numbers

$$\sum_1^n H_k y^{n-k} = \frac{by^{n+1} + ay^n - H_{n+1}\cdot y - H_n}{y^2 - y - 1} \tag{15}$$

and

$$\sum_1^n H_k 2^{n-k} = (2b + a)2^n - H_{n+3} = 2^n\cdot H_3 - H_{n+3} \tag{16}$$

which are the results obtained by King [2].

4. r-BONACCI POLYNOMIALS AS COEFFICIENTS

The r-bonacci polynomials $F_n^{(r)}(x)$ have been defined by Hoggatt and Bicknell [5] as

$$F_{-(r-2)}^{(r)}(x) = \cdots = F_{-1}^{(r)}(x) = F_0^{(r)}(x) = 0, \quad F_1^{(r)}(x) = 1, \quad F_2^{(r)}(x) = x^{r-1} ,$$

and

$$F_{n+r}^{(r)}(x) = x^{r-1}F_{n+r-1}^{(r)}(x) + x^{r-2}F_{n+r-2}^{(r)}(x) + \cdots + F_n^{(r)}(x) . \tag{17}$$

Let us now consider

$$I = \sum_1^n F_k^{(r)}(x)y^{n-k} .$$

Denoting for the sake of convenience

$$F_k^{(r)}(x) = R_k \tag{18}$$

we have,

$$I = R_1y^{n-1} + x^{r-1}R_1y^{n-2} + (x^{r-1}R_2 + x^{r-2}R_1)y^{n-3} + \cdots + (x^{r-1}R_{r-1} + x^{r-2}R_{r-2} + \cdots + xR_1)y^{n-r}$$

$$+ \sum_{r+1}^n (x^{r-1}R_{k-1} + x^{r-2}R_{k-2} + \cdots + R_{k-r})y^{n-k} = R_1y^{n-1} + x^{r-1}y^{-1}[R_1y^{n-1} + R_2y^{n-2} + \cdots + R_{r-1}y^{n-r+1}]$$

$$+ x^{r-2}y^{-2}[R_1y^{n-1} + R_2y^{n-2} + \cdots + R_{r-2}y^{n-r+2}]$$

$$\vdots$$

$$+ xy^{-(r-1)}[R_1y^{n-1}] + x^{r-1}y^{-1}\sum_{r}^{n-1} R_ky^{n-k} + x^{r-2}y^{-2}\sum_{r-1}^{n-2} R_ky^{n-k}$$

$$+ \cdots + xy^{-(r-1)}\sum_{2}^{n-r+1} R_ky^{n-k} + y^{-r}\sum_{1}^{n-r} R_ky^{n-k}\ .$$

Hence,

$$Iy^r = R_1y^{n+r-1} + (xy)^{r-1}\sum_{1}^{n-1} R_ky^{n-k} + (xy)^{r-2}\sum_{1}^{n-2} R_ky^{n-k}$$

$$+ \cdots + xy\sum_{1}^{n-r+1} R_ky^{n-k} + \sum_{1}^{n-r} R_ky^{n-k}$$

$$= R_1y^{n+r-1} + [(xy)^{r-1} + (xy)^{r-2} + \cdots + (xy) + 1]I$$

$$- (xy)^{r-1}R_n - (xy)^{r-2}\sum_{n-1}^{n} R_ky^{n-k} - \cdots$$

$$- (xy)\sum_{n-r+2}^{n} R_ky^{n-k} - \sum_{n-r+1}^{n} R_ky^{n-k}\ .$$

Thus,

$$I\left[y^r - \sum_{0}^{r-1}(xy)^k\right] = R_1y^{n+r-1} - y^{r-1}(x^{r-1}R_n + x^{r-2}R_{n-1} + \cdots + R_{n-r+1})$$
$$- y^{r-2}(x^{r-2}R_n + x^{r-3}R_{n-1} + \cdots + R_{n-r+2})$$
$$- y^{r-3}(x^{r-3}R_n + x^{r-4}R_{n-1} + \cdots + R_{n-r+3})$$
$$- \cdots - y(xR_n + R_{n-1}) - R_n\ .$$

Denoting now

$$G_n^{(r)}(x,y) = y^{n+r-1} - F_{n+1}^{(r)}(x)y^{r-1} - y^{r-2}\cdot[x^{r-2}F_n^{(r)}(x) + x^{r-3}F_{n-1}^{(r)}(x) + \cdots + F_{n-r+2}^{(r)}(x)]$$

(19)
$$- y^{r-3}[x^{r-3}F_n^{(r)}(x) + x^{r-4}F_{n-1}^{(r)}(x) + \cdots + F_{n-r+3}^{(r)}(x)]$$
$$- \cdots - y[xF_n^{(r)}(x) + F_{n-1}^{(r)}(x)] - F_n^{(r)}(x)$$

we have

(20)
$$I = \sum_{1}^{n} F_k^{(r)}(x)\cdot y^{n-k}\ \frac{G_n^{(r)}(x,y)}{G_1^{(r)}(x,y)}\ .$$

The above result for r-bonacci polynomials may be considered as a generalization of the result (7) for Fibonacci polynomials.

Let us now see if we can obtain for the r-bonacci numbers [5], a result corresponding to (10) for Fibonacci numbers; it may be noted that the r-bonacci numbers $F_n^{(r)}$ are obtained by letting $x = 1$ in (17). We have from (20) that

(21)
$$\sum_{1}^{n} F_k^{(r)}\cdot 2^{n-k} = \frac{G_n^{(r)}(1,2)}{G_1^{(r)}(1,2)}\ .$$

Now we have from (19),

$$2^{n+r-1} - G_n^{(r)}(1,2) = 2\cdot 2^{r-2}F_{n+1}^{(r)} + 2^{r-2}[F_n^{(r)} + \cdots + F_{n-r+2}^{(r)}] + 2^{r-3}[F_n^{(r)} + \cdots + F_{n-r+3}^{(r)}] + \cdots +$$
$$+ 2[F_n^{(r)} + F_{n-1}^{(r)}] + F_n^{(r)} =$$

$$= 2^{r-2}F_{n+1}^{(r)} + 2^{r-2}[F_{n+2}^{(r)}] + 2^{r-3}[F_n^{(r)} + \cdots + F_{n-r+3}^{(r)}] + \cdots + 2[F_n^{(r)} + F_{n-1}^{(r)}] + F_n^{(r)}$$

$$= 2^{r-3}[F_{n+2}^{(r)} + F_{n+1}^{(r)}] + 2^{r-3}[F_{n+2}^{(r)} + F_{n+1}^{(r)} + F_n^{(r)} + \cdots + F_{n-r+3}^{(r)}] + 2^{r-4}[F_n^{(r)} + \cdots + F_{n-r-3}^{(r)}]$$

$$+ \cdots + 2[F_n^{(r)} + F_{n-1}^{(r)}] + F_n^{(r)}$$

$$= 2^{r-3}[F_{n+3}^{(r)} + F_{n+2}^{(r)} + F_{n+1}^{(r)}] + 2^{r-4}[F_n^{(r)} + \cdots + F_{n-r+3}^{(r)}] + 2[F_n^{(r)} + F_{n-1}^{(r)}] + F_n^{(r)} .$$

Continuing the process, the above may be reduced as

$$2^{n+r-1} - G_n^{(r)}(1,2) = 2^{r-r}[F_{n+r}^{(r)} + \cdots + F_{n+1}^{(r)}] = F_{n+r+1}^{(r)} .$$

Hence,

(22) $$G_n^{(r)}(1,2) = 2^{n+r-1} - F_{n+r+1}^{(r)} .$$

Also

(23) $$G_1^{(r)}(1,2) = 2^r - \sum_0^{r-1} 2^k = 2^r - \frac{1-2^r}{1-2} = 1 .$$

Therefore from (21), (22) and (23) we get

$$\sum_1^n F_k^{(r)} \cdot 2^{n-k} = 2^{n+r-1} - F_{n+r+1}^{(r)} = 2^n \cdot F_{r+1}^{(r)} - F_{n+r+1}^{(r)} .$$

The above result may be considered as a generalization, for the r-bonacci numbers, of the result of Carlitz [1] for the Fibonacci numbers.

REFERENCES

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