RESIDUES OF GENERALIZED FIBONACCI SEQUENCES

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Consider a sequence of GF numbers, $\{w_n(b,c;P,Q)\}_{-\infty}^{\infty}$. For $b = c = 1$, L. Taylor [1] has proved the following theorem.

Theorem. The only sequences which possess the property that upon division by a (non-zero) member of that sequence, the members of the sequence leave least $+ve$, or $-ve$ residues which are either zero or equal in absolute value to a member of the original sequence are the Fibonacci and Lucas sequences.

Our objective is to consider the extension of this theorem to GF sequences by a different approach, and show that a class of sequences can be constructed to satisfy the property of this theorem in a restricted sense, i.e., for a particular member only. For convenience, $w_n(b,1;0,1)$, $w_n(b,1;2,b)$, $w_n(b,1;P,Q)$ shall be designated by u_n, v_n, H_n, respectively.

Let $H_{k+r} \equiv (-1)^{r-1}H_{k-r} \pmod{H_k}$. Assume without loss of generality, k to be $+ve$. We distinguish 2 cases: (A) $0 \leqslant r \leqslant k$, and (B) $r > k$.

(A) Evidently, the members leave least residues which are either zero or equal in absolute value to a member of the original sequence.

(B) Allow $|H_{-s+1}| \leqslant |H_k| \leqslant |H_{-s}|$. Let

$$(1) \qquad H_{2k+s} \equiv (-1)^{k+s-1}H_{-s} \pmod{H_k},\; H_{2k+s+1} \equiv (-1)^{k+s}H_{-s-1} \pmod{H_k}.$$

Clearly, the property of above-cited theorem holds for $\{H_n\}_{-\infty}^{\infty}$, iff

$$(-1)^{k+s-1}H_{-s} \equiv H_\ell \pmod{H_k}, \qquad \text{and} \qquad (-1)^{k+s}H_{-s-1} \equiv H_{\ell+1} \pmod{H_k},$$

for some ℓ such that $-s+1 \leqslant \ell \leqslant 2k$. Denote the period of $\{H_n \pmod{H_k}\}_0^{\infty}$ by $k(H_k)$. Rewrite the given sequence as $\{H'_{n'}\}_{-\infty}^{\infty}$, where $H'_{n'} = H_n$. Set $k' = k + t$, $s' = s - t$, and $\ell' = \ell + t$. Then, it is easy to show that

$$k(H_k) = 2k + s - \ell, \quad k(H'_{k'}) = 2k' + s' - \ell', \quad \text{and} \quad k(H_k) = k(H'_{k'}).$$

We assert that $k(H'_{k'})$ is even, for $t = (s-\ell)/2$ obtains $s' = \ell'$, $k(H'_{k'}) = 2k'$, and the substitution of $s - \ell = 2t + 1$ leads to $s' - \ell' = 1$, $k(H'_{k'}) = 2k' + 1$, which is a contradiction. Hence, it is sufficient to examine the following system of congruences, viz.,

$$(2) \qquad H'_{2k'} \equiv H'_0 \pmod{H'_{k'}}, \qquad H'_{2k+1} \equiv H'_1 \pmod{H'_{k'}}.$$

These congruences imply

$$(3) \qquad H_{2k+t} \equiv H_t \equiv (-1)^{k+t-1}H_{-t} \pmod{H_k} \equiv (-1)^{k-1}\{H_t - (2Q - bP)u_t\} \pmod{H_k}$$
$$\equiv (-1)^{k-1}\{Pv_t - H_t\} \pmod{H_k}.$$

Therefore, (i) $P = 0$, $Q = 1$, and (ii) $P = 2$, $Q = b$, furnish readily the desired sequences, and they are the only sequences for which the property of L. Taylor's theorem holds. For the restricted case, by using the well known formula $H_n = Pu_{n-1} + Qu_n$, it is possible to express $H_{-s} \equiv H_\ell \pmod{H_k}$, and $H_{-s-1} \equiv H_{\ell+1} \pmod{H_k}$ as two simultaneous equations in P, Q, and obtain their solution for given s, ℓ, and k. In particular, the latter case may be handled by using $k(H'_{k'}) = k(u_{k'})$, where $H'_{k'}$ is selected arbitrarily to satisfy $k' = k(u_{k'})/2$ and

$$H'_{k'} = Pu_{k'-1} + Qu_{k'},$$

determines P and Q.

Example: $\quad H'_9 = 19, \quad k(H'_9) = 18, \quad P = 9, \quad Q = -5.$

REFERENCES

1. L. Taylor, "Residues of Fibonacci-Like Sequences," *The Fibonacci Quarterly,* Vol. 5, No. 3 (Oct. 1967), pp. 298–304.
2. C. C. Yalavigi, "On a Theorem of L. Taylor," *Math. Edn.,* 4 (1970), p. 105.

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COMPOSITES AND PRIMES AMONG POWERS OF FIBONACCI NUMBERS, INCREASED OR DECREASED BY ONE

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It is well known that, among the Fibonacci numbers F_n, given by

$$F_1 = 1 = F_2, \qquad F_{n+1} = F_n + F_{n-1},$$

$F_n + 1$ is composite for each $n \geqslant 4$, while $F_n - 1$ is composite for $n \geqslant 7$. It is easily shown that $F_n^2 \pm 1$ is also composite for any n, since

$$F_n^2 \pm 1 = F_{n-2}F_{n+2}, \qquad F_n^2 \mp 1 = F_{n+1}F_{n-1}.$$

Here, we raise the question of when $F_k^m \pm 1$ is composite.

First, if $k \not\equiv 0 \pmod 3$, then F_k is odd, F_k^m is odd, and $F_k^m \pm 1$ is even and hence composite. Now, suppose we deal with $F_{3k}^m \pm 1$. Since $A^n - B^n$ always has $(A - B)$ as a factor, we see that $F_{3k}^m - 1^m$ is composite except when $(A - B) = 1$; that is, for $k = 1$. Thus,

Theorem 1. $F_k^m - 1$ is composite, $k \neq 3$.

We return to $F_{3k}^m + 1$. For m odd, then $A^m + B^m$ is known to have the factor $(A + B)$, so that $F_{3k}^m + 1^m$ has the factor $(F_{3k} + 1)$, and hence is composite. If m is even, every even m except powers of 2 can be written in the form $(2j + 1)2^i = m$, so that

$$F_{3k}^m + 1^m = (F_{3k}^{2^i})^{2j+1} + (1^{2^i})^{2j+1}$$

which, from the known factors of $A^m + B^m$, m odd, must have $(F_{3k}^{2^i} + 1)$ as a factor, and hence, $F_{3k}^m + 1$ is composite.

This leaves only the case $F_{3k}^m + 1$, where $m = 2^i$. When $k = 1$, we have the Fermat primes $2^{2^i} + 1$, prime for $i = 0, 1, 2, 3, 4$ but composite for $i = 5, 6$. It is an unsolved problem whether or not $2^{2^i} + 1$ has other prime values. We note in passing that, when $k = 2$, $F_6 = 8 = 2^3$, and $8^m \pm 1 = (2^3)^m \pm 1 = (2^m)^3 \pm 1$ is always composite, since $A^3 \pm B^3$ is always factorable. It is thought that $F_9^4 + 1$ is a prime.

Since $F_{3k} \equiv 0 \pmod{10}$, $k \equiv 0 \pmod 5$, $F_{15k'}^{2^i} + 1 = 10^{2^i} \cdot t + 1$.

Since $F_{3k}^{2^i} \equiv 6 \pmod{10}$, $i \geqslant 2$, $k \not\equiv 0 \pmod 5$, $F_{3k}^{2^i} + 1$ has the form $10t + 7$, $k \neq 0 \pmod 5$. We can summarize these remarks as

Theorem 2. $F_k^m + 1$ is composite, $k \neq 3s$, $F_{3k}^m + 1$ is composite, $m \neq 2^i$.

It is worthwhile to note the actual factors in at least one case. Since

$$F_{k+2}F_{k-2} - F_k^2 = (-1)^{k+1}$$

$$F_{k+1}F_{k-1} - F_k^2 = (-1)^k$$

moving F_k^2 to the right-hand side and then multiplying yields

$$F_{k-2}F_{k-1}F_{k+1}F_{k+2} = F_k^4 - 1.$$

We now note that

$$F_k^5 - F_k = F_{k-2}F_{k-1}F_kF_{k+1}F_{k+2}$$

which causes one to ask if this is divisible by 5!. The answer is yes, if $k \not\equiv 3 \pmod 6$, but if $k \equiv 3 \pmod 6$, then only 30 can be guaranteed as a divisor.

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